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Graphes en rubans métriques

Sous la direction de : **Vincent DELECROIX**

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Graphes en rubans métriques

Résumé : Cette thèse présente quelques contributions à l'étude des fonctions de comptage des graphes en rubans métriques. Un graphe en ruban, aussi connu sous le nom de carte combinatoire, est un plongement cellulaire d'un graphe dans une surface. On peut le représenter via le recollements de polygones ou encore via des factorisations de permutations. Une métrique sur un graphe en rubans est l'attribution d'une longueur strictement positive à chaque arête. Les fonctions de comptage donnent le nombre de graphes en rubans avec une métrique entière et combinatoire fixée (genre de la surface, degré des sommets, nombre de bords) en fonction des périmètres des bords. Notre approche à l'étude de ces fonctions est purement combinatoire et repose sur l'utilisation des bijections et chirurgies pour les graphes en rubans.

Dans un premier temps, on montre que ces fonctions sont (quasi-)polynomiales par morceaux, et on précise les régions de (quasi-)polynomialité. Ensuite, on étudie les cas où leur termes de plus haut degré sont de vrais polynômes. Notre intérêt dans ces cas vient du fait que les polynômes correspondants sont utiles pour l'énumération des surfaces à petits carreaux, qui correspondent aux points entiers des strates des surfaces de (demi-)translation (de manière équivalent, states des différentielles sur les surfaces de Riemann). Par conséquent, on peut donner des formules raffinées/alternatives pour les volumes de Masur-Veech des strates. Un exemple connu sont les polynômes de Kontsevich, qui comptent les graphes en rubans métriques trivalents de genre et périmètres des bords fixés. Ils ont été utilisés récemment par Delecroix, Goujard, Zograf et Zorich pour obtenir une formule combinatoire pour les volumes des strates principales des différentielles quadratiques.

On se concentre sur les graphes en rubans métriques face-bipartis, qui apparaissent dans l'étude des différentielles Abéliennes. On montre que pour les graphes à un sommet, les termes de plus haut degré des fonctions de comptage sur certains sous-espaces sont des polynômes explicites. En conséquence, on obtient la série génératrice des contributions des surfaces à petits carreaux à n cylindres aux volumes des strates minimales des différentielles Abéliennes, raffinant un résultat précédent de Sauvaget. Ensuite, on présente un résultat de polynomialité similaire pour les deux sous-familles de graphes qui correspondent ou composants connexes de strates minimales de parité spin paire/impaire. Cela donne un raffinement d'une formule pour les différences des volumes correspondants obtenue précédemment par Chen, Möller, Sauvaget et Zagier. Puis on conjecture que le phénomène de polynomialité reste vrai pour les familles de graphes à plusieurs sommets, si chaque graphe est pondéré par le comptage de certains arbres couvrants. On prouve cette conjecture dans le cas planaire. En chemin, on construit des familles d'arbres plans qui correspondent à certaines triangulations de produits de simplexes qui représentent un intérêt du point de vue de la théorie des polytopes. Finalement, on présente une contribution au projet commun avec Duryev et Goujard, où la formule combinatoire de Delecroix, Goujard, Zograf et Zorich est généralisée aux strates des différentielles quadratiques aux singularités impaires. La contribution est une preuve combinatoire de la formule pour les coefficients qui comptent certaines dégénérescences des graphes en ruban métriques non-face-biparti.
Mots-clés : graphes en rubans métriques, cartes combinatoires, surfaces à petits carreaux, énumération asymptotique, bijection, surfaces de translation

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Metric ribbon graphs

Abstract: This thesis presents several contributions to the study of counting functions for metric ribbon graphs. Ribbon graphs, also known as combinatorial maps, are cellular embeddings of graphs in surfaces modulo homeomorphisms. They are combinatorial objects that can be represented as gluings of polygons or factorizations of permutations. Metric on a ribbon graph is an assignment of positive lengths to its edges. The counting functions give the number of integral metric ribbon graphs with fixed combinatorics (genus of the surface, degrees of vertices, number of boundaries) as a function of the perimeters of the boundaries. Our approach to their study is purely combinatorial and relies on bijections and surgeries for ribbon graphs.

Firstly, we show that these functions are piecewise (quasi-)polynomials, specifying exactly the regions of (quasi-)polynomiality. We then study the cases when their top-degree terms are honest polynomials. Our interest in such cases comes from the fact that the corresponding polynomials can be used for refined enumeration of square-tiled surfaces, which correspond to integer points in the strata of (half-)translations surfaces (equivalently, strata of differentials on Riemann surfaces). Consequently, one can give refined/alternative formulas for Masur-Veech volumes of strata. One known example are the Kontsevich polynomials, counting trivalent metric ribbon graphs of given genus and perimeters of boundaries. They were recently used by Delecroix, Goujard, Zograf and Zorich to give a combinatorial formula for the volumes of principal strata of quadratic differentials.

We concentrate on face-bipartite metric ribbon graphs, which appear in the study of Abelian differentials. We show that in the case of one-vertex graphs the top-degree terms of the counting functions on certain subspaces are in fact (explicit) polynomials. As a consequence, we deduce the generating function for the contributions of n-cylinder square-tiled surfaces to the volumes of minimal strata of Abelian differentials, refining a previous result of Sauvaget. We then present a similar polynomiality result for the two subfamilies of graphs corresponding to even/odd spin connected components of the minimal strata. This also gives a refinement of a formula for the corresponding volume differences previously obtained by Chen, Möller, Sauvaget and Zagier. Next we conjecture that the polynomiality phenomenon holds for families of graphs with several vertices, if each graph is weighted by the count of certain spanning trees. We prove the conjecture in the planar case. In the process, we construct families of plane trees which correspond to certain triangulations of the product of two simlpices, which are interesting from the point of view of the theory of polytopes. Finally, we present a contribution to a joint work with Duryev and Goujard, where the combinatorial formula of Delecroix, Goujard, Zograf and Zorich is generalized to all strata of quadratic differentials with odd singularities. The contribution is a combinatorial proof of the formula for coefficients counting certain degenerations of (non-face-bipartite) metric ribbon graphs.
Keywords: metric ribbon graphs, combinatorial maps, square-tiled surfaces, asymptotic enumeration, bijection, translation surfaces

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Contents

Chapter 1

Introduction

This chapter contains all the necessary definitions and the background material needed to motivate and state the results of this thesis. The outline of this chapter is as follows.

In section 1.1 we define ribbon graphs, introduce the related terminology, notations and the families of ribbon graphs we are going to consider. We also give a brief overview of the study of ribbon graphs.

In section 1.2 we introduce the main objects of this thesis: metric ribbon graphs and their counting functions. We illustrate on an example the general nature of the counting functions – their piecewise (quasi-)polynomiality. We then present the case of counting functions for trivalent ribbon graphs, which are known, by a result of Kontsevich, to have a special property: their top-degree term is an honest polynomial. For completeness, we give a sketch of Kontsevich's proof. A brief overview of other works on / applications of metric ribbon graphs is given.

In section 1.3 we introduce square-tiled surfaces and show how their enumeration is related to the volumes of strata of translation surfaces (Abelian differentials). We then explain an idea from a recent paper of Delecroix, Goujard, Zograf and Zorich, where metric ribbon graphs are used for the enumeration of square-tiled surfaces. The results of this paper were the initial motivation for this thesis.

Finally, in section 1.4 we give a summary of the contributions of this thesis, which are presented in subsequent chapters.

1.1 Ribbon graphs

1.1.1 Definitions

Ribbon graphs (also known as *combinatorial maps* or *fatgraphs*) admit many equivalent definitions. We will give two of them: a topological one and a combinatorial one. We refer the reader to [LZ04] for further details (and for an excellent introduction to the topic).

Topological definition

In what follows, by a *graph* we mean a finite graph with possible loops and multiple edges . By a *surface* we mean a connected oriented compact surface without boundary. By the famous classification theorem, homeomorphism classes of such surfaces are parametrized by one non-negative integer parameter g – the *genus* of the surface. Informally, it corresponds to the number of "holes" in the surface: genus 0 surface is a sphere, genus 1 surface is a torus, genus 2 surface is a 2-holed torus, etc.

Definition 1.1 (Topological definition)**.** A *ribbon graph* is a graph embedded into a surface, in such a way that

- the vertices are represented by distinct points;
- the edges are represented by non-self-intersecting curves which can intersect only at the vertices;
- the connected components of its complement (which we call *faces*) are homeomorphic to disks.

Two ribbon graphs $G_1 \subset S_1$ and $G_2 \subset S_2$ are *isomorphic* if there is an orientation-preserving homeomorphism $f : S_1 \to S_2$ which sends the vertices and edges of G_1 to the vertices and edges of G_2. In particular, isomorphic ribbon graphs have isomorphic underlying graphs and are embedded into surfaces of the same genus.

Informally, a ribbon graph is simply a "drawing" of a connected graph on a surface. For example, a drawing of a planar graph on the plane is actually a ribbon graph on a sphere (one has to compactify the plane with one point at infinity, so that the outer face becomes a disk). Two drawings are equivalent if there is a continuous transformation of the surface sending one to the other. Note that, for a surface of genus $g \geq 1$, this transformation can be non-isotopic to the identity, a Dehn twist for example (Figure 1.1).

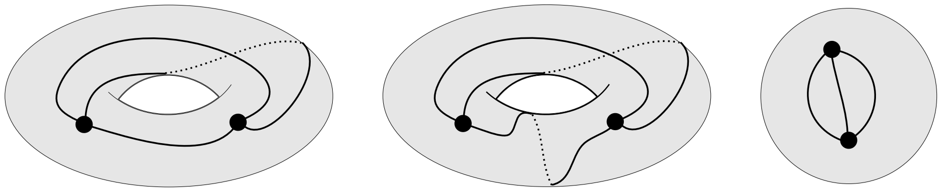


Figure 1.1: The first and the second ribbon graphs are isomorphic. The third one is not isomorphic to the first two, but their underlying graphs are.

A graph can admit embeddings into surfaces of different genera or have several nonequivalent embeddings into the same surface. Hence a single graph can give rise to multiple ribbon graphs (see Figure 1.1).

Since the surface is connected and the faces are disks, the underlying graph of a ribbon graph is necessarily connected. By a *disconnected ribbon graph* we mean a disjoint union of a finite number (at least 2) of ribbon graphs.

Combinatorial definition

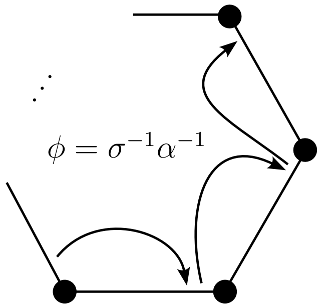
Although it might not be immediately clear from Definition 1.1, an isomorphism class of ribbon graphs is determined by a finite set of discrete data. To see this, note that an embedding of a graph into a surface induces a circular order on the edges incident to each vertex. Indeed, one can take the counterclockwise order in which the edges exit the corresponding point in the surface.

It turns out that the embedding can be recovered (up to homeomorphism) from the graph equipped with these circular orders. To this end, take an oriented rectangular ribbon for each edge (hence the name) and glue them around each vertex according to the given circular order and keeping the orientations compatible. In this way we obtain an oriented surface with boundary, which is a tubular neighborhood of the graph in the initial surface. It is then enough to glue a topological disk to each boundary to recover the surface itself.

This informal construction is made precise by the following definition.

Definition 1.2 (Combinatorial definition)**.** A *ribbon graph* is a triple (H, σ, α), where H is a finite set, σ is a permutation of H, and α is an involution of H without fixed points, such that the group generated by σ and α acts transitively on H.

In this definition H corresponds to the set of *half-edges* of the graph (each edge consists of two half-edges, each half-edge is incident to exactly



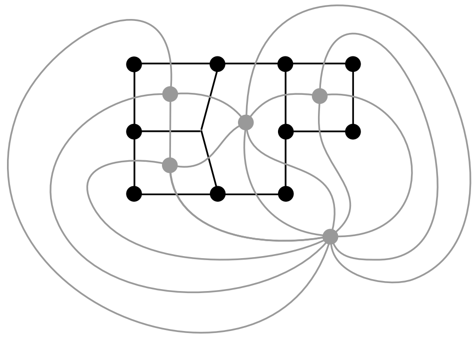


Figure 1.2: *Left*: the face permutation ϕ. *Right*: a ribbon graph (in black) and its dual (in grey).

one vertex). α sends a half-edge to the other one with which it forms an edge. σ sends a half-edge to the next (counterclockwise) half-edge around the incident vertex.

In this way, the cycles of α correspond to the edges of the ribbon graph and the cycles of σ correspond to its vertices. The transitivity condition is equivalent to the connectedness of the graph. Note that the cycles of the permutation $\phi = \sigma^{-1}\alpha^{-1}$ are in bijection with the faces of the ribbon graph (Figure 1.2, left). The three permutations σ, α, ϕ thus satisfy $\phi\alpha\sigma = \mathrm{id}$.

Naturally, two ribbon graphs $(H_1, \sigma_1, \alpha_1)$ and $(H_2, \sigma_2, \alpha_2)$ are *isomorphic* if there is a bijection $f : H_1 \to H_2$ such that $\sigma_1 = f^{-1}\sigma_2 f$ and $\alpha_1 = f^{-1}\alpha_2 f$.

Terminology and notations

Remark 1.3. *The faces of the ribbon graph are also called its* boundary components. *In this thesis we will use these two terms interchangeably.*

The sets of vertices, edges and faces of a ribbon graph G are denoted by $V(G)$, $E(G)$, $F(G)$ respectively. The *degree of a vertex* is the number of incident half-edges (hence, if there is a loop incident to a vertex, it contributes to its degree twice). An edge is *incident to a face* if it is a part of its boundary. The *degree of a face* is the number of incident edges. Note that an edge can have the same face on both sides of it, in this case this edge contributes to the degree of this face twice. The degree of a vertex v or a face f of G are denoted by $\deg_G(v)$ and $\deg_G(f)$ respectively. A *corner* is a pair of consecutive half-edges incident to the same vertex.

One clearly has

$$\sum_{v \in V(G)} \deg_G(v) = \sum_{f \in F(G)} \deg_G(f) = 2|E(G)|,$$

which is also the number of corners of G.

The *genus* of a ribbon graph G is the genus of the surface it is embedded in. It can be computed using the famous Euler's relation:

$$|V(G)| - |E(G)| + |F(G)| = 2 - 2g.$$

Ribbon graphs of genus 0 are also called *plane ribbon graphs*.

An *automorphism* of a ribbon graph $G = (H, \sigma, \alpha)$ is a permutation $f : H \to H$ such that $f\sigma f^{-1} = \sigma$ and $f\alpha f^{-1} = \alpha$. In the topological definition, this corresponds to a homeomorphism of the surface sending the ribbon graph onto itself. The automorphisms of G clearly form a group, which we denote by $\mathrm{Aut}(G)$.

Remark 1.4. *From now on, when depicting a ribbon graph, we will only show its vertices and edges, omitting the underlying surface. The cyclic order of edges in a small neighborhood of each vertex is the one prescribed by the ribbon graph structure. Outside of these neighborhoods, the edges are drawn in an arbitrary manner.*

Coloring and labeling

We say that a ribbon graph is *bipartite* if its underlying graph is bipartite. A ribbon graph is *vertex-bicolored* if its vertices are colored in black and white in such a way that adjacent vertices have different colors.

A ribbon graph is *face-bicolored* if its faces are colored in black and white in such a way that for any edge, the two faces incident to it are of different colors. A ribbon graph admitting such coloring is called *face-bipartite*.

In this thesis, the faces and/or the vertices of the ribbon graphs will often carry distinct (usually positive integer) labels.

Remark 1.5. *If we consider ribbon graphs with coloring and/or labeling, the isomorphisms between them should preserve these additional data.*

Rooting

A ribbon graph is *rooted* if it has a distinguished corner, the *root corner*. In figures we denote the root corner by an additional oriented *root half-edge* pointing to it. The incident vertex is called the *root vertex*. The edge of G following the root half-edge *clockwise* around the corresponding vertex is called the *root edge*. The *root face* is the face containing the root corner.

Two rooted ribbon graphs are isomorphic if there is an isomorphism of the underlying ribbon graphs which sends the root corner to the root corner.

Note also that any automorphism of a ribbon graph preserving the rooting is necessarily the identity (i.e. rooted ribbon graphs have no symmetries).

Vertex-bicolored ribbon graphs should always be rooted at a black vertex. Face-bicolored ribbon graphs should always be rooted at a black face.

Dual ribbon graphs

We now associate to any ribbon graph its *dual ribbon graph*. We first give a topological construction.

Let G be a ribbon graph. Put a new vertex inside each face of G. Then, for each edge e of G, join the two new vertices corresponding to the two faces of G incident to e by a new edge e^* that only intersects e (Figure 1.2, right). Note that if e was incident twice to the same face, then e^* is a loop. The ribbon graph formed by the new vertices and the new edges is the dual ribbon graph G^*.

One can easily see that the dual of G^* is (isomorphic to) G. The vertices and the faces of G correspond bijectively to the faces and the vertices of G^* respectively. There is also a bijective correspondence between their edges. For $e \in E(G)$ we denote by $e^* \in E(G^*)$ the corresponding edge of the dual graph.

In terms of permutations, if $G = (H, \sigma, \alpha)$, then the dual graph is given by $G^* = (H, \phi, \alpha)$, where $\phi = \sigma^{-1}\alpha^{-1}$ is the face permutation of G.

Note also that any coloring/labeling of vertices or faces of G induces a coloring/labeling of the corresponding faces or vertices of G^* respectively. In particular, if G is face-bicolored, then G^* is vertex-bicolored.

1.1.2 Families of ribbon graphs

We define here the families of ribbon graphs we are going to consider in this thesis. Before that, we set up some notation.

Compositions and partitions

Recall that a *composition of length* n of a positive integer N is sequence of positive integers $(d_1, \ldots, d_n)$ with sum N. A *partition of length* n of a positive integer N is a sequence of positive integers $(k_1, \ldots, k_n)$ with sum N and such that $k_1 \geq k_2 \geq \ldots \geq k_n$. N is called the *weight* of the composition/partition.

To distinguish between compositions and partitions, we will use parentheses to denote the former and square brackets to denote the latter. For example, $(1, 2, 5, 1, 4)$ is a composition of 13, while $[5, 4, 2, 1, 1]$ is a partition of 13. Compositions are denoted by simple letters: $d = (d_1, \dots, d_n)$. Partitions are denoted by underlined letters: $\underline{k} = [k_1, \dots, k_n]$.

Finally, we will also use an abbreviated representation of partitions: $\underline{k} = [1^{\alpha_1}, 2^{\alpha_2}, \dots]$ if there are α_1 parts equal to 1 in $\underline{k}$, α_2 parts equal to 2 in $\underline{k}$, etc. Parts with $\alpha_i = 0$ are omitted, as are the exponents $\alpha_i = 1$. For example, $[5, 3, 3, 2, 1, 1, 1] = [5, 3^2, 2, 1^3]$.

Face-bicolored families

For $g \geq 0$, $k, l \geq 1$, denote by $\mathcal{RG}_{g,(k,l)}$ the set of isomorphism classes of ribbon graphs of genus g which are face-bicolored, with k black faces labeled from 1 to k, l white faces labeled from 1 to l.

For a composition $d = (d_1, \dots, d_n)$, we denote by $\mathcal{RG}^{d}_{g,(k,l)}$ the family of face-bicolored ribbon graphs of genus g with k black and l white labeled faces, and n *labeled* vertices of corresponding degrees $d_1, \dots, d_n$. Such graphs are studied in Chapter 6.

In Chapters 3 and 4 we study face-bicolored ribbon graphs of genus g with k black and l white labeled faces and *with one vertex*. The sets of such graphs will be denoted by $\mathcal{E}_{g,k,l}$.

Non-face-bicolored families

For $g \geq 0$, $n \geq 1$, denote by $\mathcal{RG}_{g,n}$ the set of isomorphism classes of ribbon graphs of genus g, with n faces labeled from 1 to n.

For a partition $\underline{k} = [k_1, \dots, k_s]$ we denote by $\mathcal{RG}^{\underline{k}}_{g,n} \subset \mathcal{RG}_{g,n}$ the subset of ribbon graphs with s vertices of degrees $k_1, \dots, k_s$ (the vertices are *unlabeled* here). Such graphs are considered in Chapter 7.

Dual and rooted families

For any of the families defined above, the corresponding family of dual ribbon graphs will be denoted by a star in the superscript. For example, $\mathcal{RG}^{d,*}_{g,(k,l)}$ is the set of ribbon graphs of genus g, which is vertex-bicolored, with k black vertices labeled from 1 to k, l white vertices labeled from 1 to l, and n labeled faces of degrees $d_1, \dots, d_n$. Note also that the graphs in $\mathcal{E}^*_{0,k,l}$ are vertex-bicolored plane trees.

Finally, the corresponding sets of rooted ribbon graphs are denoted by the word "root" in the superscript: $\mathcal{RG}^{d,*,root}_{g,(k,l)}$.

These notations will be reminded when needed throughout the text.

1.1.3 Brief overview of ribbon graphs

Analytic and algebraic approaches

Enumeration of ribbon graphs is a classic topic in combinatorics dating back to the works of Tutte on planar ribbon graphs in the 60's [Tut62b, Tut63, Tut62c, Tut62a]. One of the famous results obtained using his *quadratic method* is his formula for the number of rooted plane ribbon graphs with n edges [Tut63]:

$$\frac{2 \cdot 3^n (2n)!}{n!(n+2)!}.$$

The idea of Tutte's quadratic method is to use a simple recursive decomposition of ribbon graphs (which amounts to deleting one edge) to obtain a (quadratic, difference) equation for the corresponding generating series. However, to even be able to write down such equation, one has to introduce an additional parameter (the degree of the root face), and to consider the corresponding *bivariate* generating series. The additional variable in this series is commonly referred to as the *catalytic variable*, since its introduction is necessary to find the corresponding univariate series. One is then able to solve this equation (with some work) and obtain the explicit formula for its coefficient.

Tutte's method is rather general and can be applied to different families of maps: triangulations [Tut62b], maps of positive genus [BC86], etc. Equations with a catalytic variable arising in such problems are now well-understood due to the work of Bousqet-Mélou and Jehanne [BMJ06].

By using techniques of *singularity analysis* of generating series [FS09], one can also obtain asymptotic enumerations results. For example, Bender and Canfield [BC86] have shown that the number of ribbon graphs of genus g with n edges grows as

$$t_g n^{5(g-1)/2} 12^n$$

when g is fixed and $n \to \infty$, where t_g is a computable constant. It has been shown later then many families of ribbon graphs have a very similar asymptotic enumeration behaviour, a phenomenon known as *universality*.

It has been recently understood that Tutte's recursions for ribbon graphs are particular cases of *topological recursion* [Eyn16]. It is a general recursive framework satisfied by many different problems related to enumeration of structures on surfaces, which has its roots in random matrix theory.

The connection between enumeration of ribbon graphs and *matrix integrals* was known to physicists [Hoo74] and was rediscovered by mathematicians in the paper of Harer and Zagier [HZ86], where they compute the Euler characteristic of the moduli space of curves. In particular, they prove that the (properly defined) bivariate generating series of ribbon graphs of genus g, with n edges and one face has the form

$$\left(\frac{1+x}{1-x}\right)^y.$$

Recall that ribbon graphs admit a definition in terms of permutations (Definition 1.2). It turns out that enumeration of ribbon graphs is equivalent to the enumeration of factorizations of permutations satisfying certain properties. Such problems can be approached using the representation theory of the symmetric groups, see [LZ04, Appendix A] for a short introduction to the topic. The main tool here is the famous Frobenius formula expressing the number of factorizations of the identity into elements of fixed conjugacy classes in terms of the characters of the irreducible representations. In certain cases it allows to get explicit formulas. For example, Goupil and Schaeffer obtained in [GS98] explicit formulas for the numbers of bipartite ribbon graphs with one face and fixed vertex degrees. Another famous result is the proof by Goulden and Jackson [GJ08] that the generating series of ribbon graphs satisfies the KP hierarchy, which is a certain integrable hierarchy of PDEs coming from mathematical physics.

Ribbon graphs can also be seen as *ramified covers* of the sphere, see [LZ04, Chapters 1,5]. The numbers of isomorphism classes of covers with prescribed ramification profiles over fixed points are known as *Hurwitz numbers*. Since ramified covers of the sphere are equivalent to meromorphic functions on Riemann surfaces, the study of Hurwitz numbers is closely related to algebraic geometry of the moduli spaces of curves. One of the famous results here is the *ELSV formula*, due to Ekedahl, Lando, Shapiro and Vainshtein [ELSV01], which concerns *simple Hurwitz numbers* $h_{g;k_1,\ldots,k_n}$. These numbers count covers of genus g with one ramification profile equal to $(k_1, \ldots, k_n)$ and all other ramification profiles being simple (of the form $2, 1, \ldots, 1$). The ELSV formula expresses the (normalized) simple Hurwitz numbers as polynomials in k_i whose coefficients are certain intersection numbers on the moduli space of curves $\mathcal{M}_{g,n}$. Similar results have since been obtained for many other families of Hurwitz numbers.

Bijective approaches

In another direction, a variety of bijections between different families of ribbon graphs and decorated trees have been discovered. Such bijections often give elegant proofs of enumerative formulas previously obtained by algebraic methods. More importantly, they allow to study certain fine properties of ribbon graphs.

One of the most famous bijections is the Cori-Vauquelin-Schaeffer bijection between planar quadrangulations and properly labeled plane trees [CV81], [Sch98]. It has since been generalized to general Eulerian planar ribbon graphs by Bouttier, Di Francesco and Guitter [BDFG04], and to ribbon graphs on positive genus surfaces by Chapuy, Marcus and Schaeffer [CMS09]. An important property of these bijections is that they allow to control the geodesic distances in the initial ribbon graph via the decorations of the corresponding tree. This approach allowed Chassaing and Schaeffer to prove in [CS04] that the diameter of a random planar quadrangulation with n edges grows as $n^{1/4}$ when $n \to \infty$. This has initiated the asymptotic study of random ribbon graphs, culminating in the independent proofs by Le Gall [LG13] and Miermont [Mie13] of the convergence of uniform planar maps to the *Brownian sphere*. The Brownian sphere is a random metric space of Hausdorff dimension 4 which is, however, almost surely homeomorphic to a sphere. The convergence is in the Gromov-Hausdorff sense, where a quadrangulation is considered as a metric space with its graph distance (which should be scaled by $n^{-1/4}$). Since then many families of ribbon graphs have been shown to converge to the Brownian sphere, confirming the universality phenomenon.

Other well-known bijections include the bijections between ribbon graphs and blossoming trees due to Schaeffer [Sch98], between ribbon graphs with one boundary and trees decorated by permutations due to Chapuy-Féray-Fusy [CFF13], bijections for planar ribbon graphs with a distinguished spanning tree due to Bernardi [Ber07] and its generalization to ribbon graphs with a distinguished spanning unicellular subgraph due to Bernardi and Chapuy [BC11].

1.2 Metric ribbon graphs

1.2.1 Definitions

Definition 1.6. A *metric* on a ribbon graph is an assignment of a positive number (*length*) to each each edge. A metric is *integer* if all the lengths are integer numbers.

A *metric ribbon graph* is simply a ribbon graph equipped with a metric. Given a metric ribbon graph, the *perimeter of a boundary component* is simply the sum of lengths of edges incident to this boundary component (if an edge is incident to the face twice, then its length contributes twice to the perimeter).

Counting functions

Suppose $G \in \mathcal{RG}_{g,n}$ (in particular, the boundary components of G are labeled). For $L = (L_1, \dots, L_n) \in \mathbb{Z}^n$ denote by $\mathcal{N}_G(L)$ the number of integer metrics on G with perimeter of the boundary component with label i equal to L_i, for each $i = 1, \dots, n$.

Likewise, for $G \in \mathcal{RG}_{g,(k,l)}$ and for any $(L, L') = (L_1, \dots, L_k; L'_1, \dots, L'_l) \in \mathbb{Z}^k \times \mathbb{Z}^l$ denote by $\mathcal{P}_G(L, L')$ the number of integer metrics on G with perimeters of the corresponding black and white boundary components being $L_1, \dots, L_k$ and $L'_1, \dots, L'_l$ respectively.

We call $\mathcal{N}_G$ or $\mathcal{P}_G$ the *counting function of* G.

For $g \geq 0$, $n \geq 1$ and $\underline{k}$ a partition, we define the *counting function of the family* $\mathcal{RG}^{\underline{k}}_{g,n}$ as the weighted sum

$$\mathcal{N}^{\underline{k}}_{g,n}(L) = \sum_{G \in \mathcal{RG}^{\underline{k}}_{g,n}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{N}_G(L).$$

For $g \geq 0$, $k, l \geq 1$ and d a composition, the *counting function of the family* $\mathcal{RG}^{d}_{g,(k,l)}$ is defined analogously as

$$\mathcal{P}^{d}_{g,(k,l)}(L, L') = \sum_{G \in \mathcal{RG}^{d}_{g,(k,l)}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L, L').$$

In Chapters 3 and 4 we are going to study face-bicolored ribbon graphs with one vertex. For brevity, the counting function of the corresponding family will be denoted by $\mathcal{P}^{g}_{k,l}(L, L')$.

1.2.2 Metrics on dual ribbon graphs

Suppose given a metric on a ribbon graph G. It induces a metric on its dual G^* by assigning to each dual edge e^* the length equal to the length of e. The perimeter of each boundary component of G is then equal to the sum of lengths of edges incident to the corresponding vertex of G^*. By analogy, we call these numbers the *perimeters of vertices of* G^*.

Convention 1.7. *In our proofs, we will mostly work with the dual families of ribbon graphs. Instead of counting metrics on ribbon graphs with given perimeters of the boundary components, we will count metrics on their dual ribbon graphs with given vertex perimeters.*

For any $G \in \mathcal{RG}^*_{g,n}$ *we define the counting function* $\mathcal{N}_G(L)$ *as the number of integer metrics on* G *with vertex perimeters given by* L.

For any $G \in \mathcal{RG}^*_{g,(k,l)}$ *we define the counting function* $\mathcal{P}_G(L, L')$ *as the number of integer metrics on* G *with vertex perimeters given by* L, L'.

Hence, for $G \in \mathcal{RG}_{g,n}$ we have $\mathcal{N}_G(L) = \mathcal{N}_{G^*}(L)$, and for $G \in \mathcal{RG}_{g,(k,l)}$ we have $\mathcal{P}_G(L, L') = \mathcal{P}_{G^*}(L, L')$.

The counting functions $\mathcal{N}^{\underline{k}}_{g,n}(L)$ and $\mathcal{P}^d_{g,(k,l)}(L, L')$ admit then alternative definitions using dual ribbon graphs:

$$\mathcal{N}^{\underline{k}}_{g,n}(L) = \sum_{G \in \mathcal{RG}^{\underline{k},*}_{g,n}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{N}_G(L),$$

and

$$\mathcal{P}^d_{g,(k,l)}(L, L') = \sum_{G \in \mathcal{RG}^{d,*}_{g,(k,l)}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L, L').$$

1.2.3 Example of computation of a counting function

Let $G \in \mathcal{RG}_{0,(2,2)}$ be the face-bicolored plane ribbon graph in Figure 1.3. The black and white boundary components are depicted with dark and light grey respectively. We would like to compute its counting function $\mathcal{P}_G(L_1, L_2; L'_1, L'_2)$.

First of all, it is clear that $\mathcal{P}_G$ is zero if at least one of L_i or L'_j are non-positive. It is also zero if the condition $L_1 + L_2 = L'_1 + L'_2$ is not satisfied. Indeed, since the graph is face-bicolored, each edge contributes its length to both the sum of black and the sum of white perimeters, so these sums must be equal.

Suppose now that L_i, L'_j are positive and $L_1 + L_2 = L'_1 + L'_2$. Let a, b, c, d be the lengths of the edges of G as in Figure 1.3. Then, clearly, $\mathcal{P}_G(L_1, L_2; L'_1, L'_2)$ is the number of integer solutions to the following system of linear equations and inequalities:

$$\begin{cases} a, b, c, d > 0 \\ a = L_1 \\ b + c + d = L_2 \\ a + b = L'_1 \\ c + d = L'_2 \end{cases}$$

First and third equations uniquely determine the values of a and b: $a = L_1$, $b = L'_1 - L_1$. The second equation is equivalent to the fourth: $c + d = L_2 - b = L_2 - (L'_1 - L_1) = L'_2$. The number of positive integer solutions to the fourth is clearly equal to $L'_2 - 1$. Hence, one could think that the total count is simply $L'_2 - 1$. But notice that $b = L'_1 - L_1$ is positive only when $L'_1 > L_1$. Hence the actual answer is

$$\mathcal{P}_G(L_1, L_2; L'_1, L'_2) = \mathbf{1}_{L'_1 > L_1} \cdot (L'_2 - 1),$$

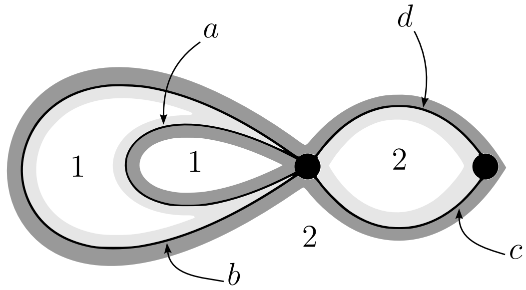


Figure 1.3: Computing the counting function of a face-bicolored ribbon graph.

where **1** denotes the indicator function.

One can already see on this example the nature of the counting functions of ribbon graphs – they are *piecewise polynomials*, whose regions of polynomiality are delimited by the hyperplanes of the form $\sum_{i\in I} L_i = \sum_{j\in J} L'_j$. This property will be explained and proved in full generality in Chapter 2 (with a slight modification in the case of non-face-bipartite ribbon graphs: the corresponding counting functions will turn out to be piecewise *quasi*-polynomials).

Consequently, the counting function of any *family* of ribbon graphs is also piecewise (quasi-)polynomial. The regions of (quasi-)polynomiality for different ribbon graphs are in general different and the number of these regions grows exponentially as a function of the number of boundary components. Hence, a priori, the counting functions for families of ribbon graphs are hard to compute explicitly. However, there are cases where simplifications occur. One such case is given by the families of trivalent ribbon graphs, presented in the next section.

1.2.4 Kontsevich polynomials

Let $g \geq 0, n \geq 1$. Consider the set of ribbon graphs of genus g with n labeled boundary components and which are trivalent, i.e. all vertices have degree 3. A simple computation using Euler's formula shows that the number of vertices is then equal to $4g + 2n - 4$. Hence, in our notations, this family of ribbon graphs is

$$\mathcal{RG}_{g,n}^{[3^{4g+2n-4}]}.$$

Recall that $\mathcal{N}_{g,n}^{[3^{4g+2n-4}]}$ denotes the counting function of this family. The following result is part of the proof by Kontsevich [Kon92] of the famous Witten's conjecture [Wit91].

Theorem 1.8 ([Kon92])**.** *Fix $g \geq 0$ and $n \geq 1$. There exists a homogeneous polynomial $N_{g,n}$ in the variables L_i^2 of degree $6g-6+2n$ such that for all $L \in \mathbb{Z}_{>0}^n$ such that $\sum_{i=1}^n L_i$ is even, we have*

$$\mathcal{N}_{g,n}^{[3^{4g+2n-4}]}(L) = N_{g,n}(L) + \text{terms of lower degree.}$$

The coefficients of $N_{g,n}$ are certain intersection numbers on the moduli space of genus g curves with n marked points.

What this theorem says is that, despite the piecewise nature of the counting function of each graph, the top-degree term of their weighted sum does not posses discontinuities – it is an honest polynomial. The precise formal sense of "terms of lower degree" is of course that they are piecewise quasi-polynomials of degree less then $6g-6+2n$.

It was shown by Norbury [Nor10] that the counting function for the family of *all* ribbon graphs of genus g, with n boundary components and vertex degrees at least 3, is a (non-piecewise) quasi-polynomial. Its top-degree term coincides with the one of $\mathcal{N}_{g,n}^{[3^{4g+2n-4}]}$. The graphs with at least one vertex degree bigger than 3 contribute to the lower-degree terms. The integer metric ribbon graphs counted by this function correspond to the lattice points in the moduli spaces of curves (see section 1.2.5 for more information on the connection with moduli spaces).

We call $N_{g,n}$ the *Kontsevich polynomials*. We now define the intersection numbers that are the coefficients of $N_{g,n}$.

Let $\mathcal{M}_{g,n}$ denote the moduli space of compact Riemann surfaces (algebraic curves) of genus g with n marked points. Its elements are tuples $(C, x_1, \ldots, x_n)$, where C is Riemann surface of genus g, and x_i are distinct labeled points of C. $\mathcal{M}_{g,n}$ is a complex orbifold of complex dimension $3g-3+n$. Over $\mathcal{M}_{g,n}$ there are n *tautological line bundles* $\mathcal{L}_i$, $i = 1, \ldots, n$. The fiber of $\mathcal{L}_i$ over a point $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ is the cotangent line $T_{x_i}^* C$. Let $\psi_i \in H^2(\mathcal{M}_{g,n}; \mathbb{Q})$ denote the first Chern class of the bundle $\mathcal{L}_i$. Define the following intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $d_i \geq 0$ and $d_1 + \ldots + d_n = 3g-3+n$. Kontsevich shows that these numbers are in fact the coefficients of the polynomials $N_{g,n}$:

$$N_{g,n}(L) = \frac{1}{2^{5g-6+2n}} \sum_{d_1+\ldots+d_n=3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^n \frac{L_i^{2d_i}}{d_i!}. \tag{1.1}$$

For completeness, we explain the idea of Kontsevich's proof of this fact in the next section.

Witten's conjecture

Theorem 1.8 and relation (1.1) are part of Kontsevich's proof of Witten's conjecture. This conjecture asserts, based on physical intuition, that the generating series of the intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ for all g, n, satisfies the KdV hierarchy – an integrable hierarchy of PDEs coming from mathematical physics. After establishing Theorem 1.8 and (1.1), Kontsevich uses the relation between ribbon graph enumeration and matrix integrals to expresses the desired generating series as an asymptotic expansion of a certain matrix integral. The KdV equations are then deduced from the analysis of this integral. Note that certain technical points of Kontsevich's proof were later addressed in papers [Loo95] and [Zvo04]. Several other proofs of Witten's conjecture have appeared since the paper of Kontsevich [AO01], [Mir07], [KL07], [ABC+21].

1.2.5 Idea of proof of Theorem 1.8

For completeness, we give here a sketch of the proof of Theorem 1.8. For more details (and an accessible exposition) see [LZ04, Chapter 4]. See also the original papers [Kon92] and [Zvo04].

Combinatorial moduli spaces

The first step to prove Theorem 1.8 and relation (1.1) is to construct a "combinatorial model" for the moduli space $\mathcal{M}_{g,n}$ using metric ribbon graphs. More precisely, we will construct a cellular complex $\mathcal{M}_{g,n}^{comb}$ called the *combinatorial moduli space* which will turn out to be homeomorphic to the *decorated moduli space of curves* $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0}$.

For any $G \in \mathcal{RG}_{g,n}$ consider the cone $\mathbb{R}_{\geq 0}^{E(G)}$. The interior points of this cone clearly correspond to metrics on G. The points in the boundary of this cone correspond to "metrics" where some of the edges are of length zero. Any such "metric" can be identified with a valid metric on another ribbon graph G', obtained by contracting the zero-length edges of G. This metric corresponds to a point in the interior of the cone $\mathbb{R}_{\geq 0}^{E(G')}$.

The cells of the combinatorial moduli space $\mathcal{M}_{g,n}^{comb}$ are the cones $\mathbb{R}_{\geq 0}^{E(G)}$ for all $G \in \mathcal{RG}_{g,n}$ with vertex degrees at least 3 (one can check that, for fixed g, n, there is only a finite number of such ribbon graphs). The cells are glued together according to the identifications described above.

The constructed space $\mathcal{M}_{g,n}^{comb}$ clearly parameterizes metric ribbon graphs of genus g with n labeled boundaries and vertex degrees at least 3. It is a real orbifold of dimension $6g - 6 + 3n$ (the orbifold points come from graphs

G with non-trivial automorphism groups, we omit the details). The highest dimensional cells correspond to trivalent metric ribbon graphs (using Euler's formula one can check that such graphs indeed have $6g-6+3n$ edges). The graphs obtained from a trivalent graph by contracting M edges represent cells of codimension M.

Jenkins-Strebel differentials

We now construct a map from the *decorated moduli space of curves* $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0}$ to the combinatorial moduli space $\mathcal{M}^{comb}_{g,n}$.

Recall that a *meromorphic quadratic differential* on a Riemann surface C is a meromorphic section of the square of the cotangent bundle $(T^*C)^{\otimes 2}$. In any holomorphic local coordinate z on C, a quadratic differential has the form $f(z)dz^2$ with f a meromorphic function. We will only consider differentials with poles of order 2.

Any non-zero quadratic differential defines a flat metric on the complement of its zeros and poles in C. In local coordinates this metric is given by $|f(z)||dz|^2$. A *horizontal trajectory* of a quadratic differential is a curve along which $f(z)dz^2$ is real and positive. In the corresponding flat metric these trajectories are represented by geodesic lines.

A *Jenkins-Strebel* differential is a quadratic differential such that all but a finite number of horizontal trajectories are closed. The non-closed trajectories necessarily start and end at the zeros of the differential. Their lengths in the induced flat metric are finite. Near a zero of order k, there is always a local coordinate z in which the differential has the form $z^k dz^2$. In particular, there are $k+2$ horizontal trajectories emanating from this point. Thus the non-closed trajectories form a (possibly disconnected) metric ribbon graph with vertex degrees at least 3.

The complement of this ribbon graph is foliated by the closed trajectories. Consequently, the connected components of the complement are either annuli, or punctured disks with a double pole at the punctured point. The lengths of all trajectories in a single component are the same. In the induced flat metric these components are represented by flat cylinders: of finite height for annuli, and infinite in one direction for punctured disks.

The following theorem is due to Strebel.

Theorem 1.9 ([Str84])**.** *Let g, n be such that $n > 2-2g$. Then for any $(C, x_1, \ldots, x_n) \in \mathcal{M}_{g,n}$ and any $(L_1, \ldots, L_n) \in \mathbb{R}^n_{>0}$ there exists a unique Jenkins-Strebel quadratic differential on C such that the closed trajectories form n punctured disks, with the punctured points being x_i, and such that the lengths of all trajectories around the point x_i are equal to L_i.*

For such differentials, the corresponding metric ribbon graph is necessarily connected (since its faces are topological disks). Note that the perimeters of its boundary components are clearly equal to $L_1, \ldots, L_n$.

We can thus define the map $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to \mathcal{M}^{comb}_{g,n}$ which associates to $(C, x_1, \ldots, x_n)$ and $(L_1, \ldots, L_n)$ the metric ribbon graph formed by the non-closed trajectories of the Jenkins-Strebel differential given by Theorem 1.9.

The payoff of this construction is the following

Proposition 1.10 ([Kon92]). *The map $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to \mathcal{M}^{comb}_{g,n}$ constructed above is a homeomorphism of real orbifolds.*

Combinatorial analogues of ψ classes

On each highest dimensional cell of $\mathcal{M}^{comb}_{g,n}$ corresponding to a trivalent ribbon graph G define a differential 2-form ω_i as follows. Choose a counterclockwise numbering of edges incident to the face with label i: $e_1, \ldots, e_s$. Let $l_{e_1}, \ldots, l_{e_s}$ be the lengths of these edges and let L_i be the perimeter of this face. Then

$$\omega_i = \sum_{1 \le j < k \le s} d(l_{e_j}/L_i) \wedge d(l_{e_k}/L_i).$$

Kontsevich shows that these explicit 2-forms represent the pushforward of the cohomology classes ψ_i via the homeomorphism $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to \mathcal{M}^{comb}_{g,n}$.

Let $\pi : \mathcal{M}^{comb}_{g,n} \to \mathbb{R}^n_{>0}$ be the map associating to a metric ribbon graph the perimeters of its boundary components. By the construction of the previous section, the composition of $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to \mathcal{M}^{comb}_{g,n}$ with π is simply the projection on the second term.

The final step of the proof is to compute the volumes of the fiber $\pi^{-1}(L)$ with respect to two different volume forms. The first one is induced by the symplectic form $\Omega = \sum_{i=1}^n L_i^2 \omega_i$ and is equal to $\Omega^{3g-3+n}/(3g-3+n)!$. The second one is the quotient Lebesgue volume form $(\prod_{e \in E(G)} dl_e)/(\prod_{i=1}^n dL_i)$.

The first volume is shown to be equal to

$$\sum_{d_1+\ldots+d_n=3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^n \frac{L_i^{2d_i}}{d_i!}.$$

The second one is the top-degree term of the counting function $\mathcal{N}^{[3^{4g+2n-4}]}_{g,n}(L)$. Indeed, the trivalent integer metric ribbon graphs with perimeters of the boundary components given by L represent the integer points in $\pi^{-1}(L)$, and so their count is closely related to the Lebesgue measure.

Kontsevich concludes by showing that the ratio of the two volume forms is a constant equal to $2^{5g-6+2n}$ on each cell, and so the corresponding volumes also differ by this factor.

1.2.6 Brief overview of metric ribbon graphs

The Kontsevich polynomials $N_{g,n}(L)$ appear also as the top-degree terms of the polynomials giving the *Weil-Petersson volumes* of the moduli space of hyperbolic surfaces with geodesic boundaries of length $L_1, \ldots, L_n$ obtained by Mirzakhani [Mir07].

This coincidence is not accidental. In the paper [ABC⁺21] a complete parallel between the geometries of the combinatorial and hyperbolic Teichmüller/moduli spaces was developed. It allowed in particular to obtain a geometric proof of Witten's conjecture by reproducing in combinatorial context Mirzakhani's proof of her recursion for the Weil-Petersson volumes.

The homeomorphism $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to \mathcal{M}^{comb}_{g,n}$ is due to Mumford and originates from the works of Jenkins [Jen57] and Strebel [Str67]. Apart from Kontsevich's proof of Witten's conjecture, it has also been used by Harer and Zagier to compute the Euler characteristic of $\mathcal{M}_{g,n}$ [HZ86].

There exists another homeomorphism $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to \mathcal{M}^{comb}_{g,n}$ due to Penner [Pen88] and Bowditch-Epstein [BE88] which uses the *spine construction* in hyperbolic geometry. More precisely, for each $L \in \mathbb{R}^n_{>0}$, the moduli space of hyperbolic surfaces of genus g with n geodesic boundaries of lengths given by L is homeomorphic to $\mathcal{M}_{g,n}$. For any such hyperbolic surface the spine is defined as the locus of points for which the distance to the boundary of the surface is realised by at least two geodesics. This locus is a ribbon graph with vertices of degrees at least 3. Each edge is given a length which is equal to the hyperbolic length of its projection to (one of the two) closest boundary of the surface. The boundary perimeters of this metric are then given by L.

Using this homeomorphism one can show that the spine metric ribbon graph of the surface is a certain scaling limit of the corresponding hyperbolic surface. To make this precise, define the *rescaling flow* ϕ_t as: applying the spine construction, rescaling the lengths of the ribbon graph by t and applying the inverse of the spine construction. If S is a hyperbolic surface with boundaries of lengths given by L, then $\phi_t S$ has boundaries of lengths given by $t \cdot L$. It has been shown in the works of Mondello [Mon09] and Do [Do10] that the hyperbolic surface $t^{-1} \cdot \phi_t S$ converges in Gromov-Hausdorff topology to the spine metric ribbon graph of S.

Recently, the *length spectrum* of *random* metric ribbon graphs in $\mathcal{M}^{comb}_{g,n}$ with fixed boundary perimeters has been studied by Barazer-Giacchetto-Liu [BGL23], generalizing the previous work of Janson-Louf on random ribbon graphs with one face [JL23]. The length spectrum of a metric ribbon graph is the multiset of lengths of closed simple cycles (each edge is traversed at most once). The above papers show that, in the large genus limit, the length spectrum converges in distribution to the Poisson point process on $\mathbb{R}_{>0}$ with

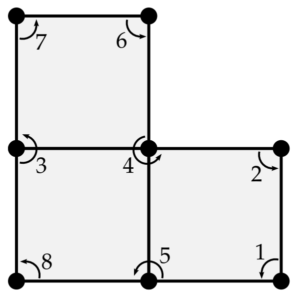


Figure 1.4: Identifying the opposite sides of this non-convex octagon, one obtains a surface of genus 2 tiled with 3 squares. All of the vertices of the squares are identified together into one conical singularity of angle $12 \cdot \frac{\pi}{2} = 6\pi$. The numbered arrows represent the order in which the corners are visited when going around the singularity.

intensity $\lambda(l) = (\cosh(l) - 1)/l$. Interestingly, the length spectra of random hyperbolic surfaces have been shown by Mirzakhani-Petri [MP19] to converge in large genus limit to the Poisson point process with *exactly* the same intensity. The methods of [JL23] are purely combinatorial, while [BGL23] uses the framework of [ABC$^+$21].

1.3 Square-tiled surfaces

1.3.1 Definition

A *square-tiled surface* is a closed oriented surface constructed from a finite collection of oriented unit squares $[0, 1]^2$ by isometrically identifying their sides respecting the orientation and the following gluing rule:

> sides are only identified in pairs of type top/bottom or left/right.

An example of a square-tiled surface is given in Figure 1.4.

The vertices and the sides of squares clearly form a ribbon graph embedded into the resulting surface, with all faces of degree 4 (i.e. this is a *quadrangulation*). Note also that the gluing rule implies that the degrees of all vertices are divisible by 4. However, this condition does not characterize completely the quadrangulations that can obtained in this way. The restriction imposed by the gluing rule can be concisely formulated in geometric terms.

Equip each square $[0, 1]^2$ with its standard flat metric. Since the sides of the squares are identified by isometries, the induced metric on the surface is also flat in the neighborhood of any point belonging to a side of a square, as well as a vertex of degree 4. Hence the square-tiled surface inherits a flat metric on the complement of the *conical singularities* – vertices of degree bigger then 4 (the total angle around a vertex of degree $4k$ is equal to $4k \cdot \pi/2 = 2\pi k$). The restriction is then equivalent to the fact that

the holonomy of this singular flat metric is trivial,

i.e. parallel transport of any tangent vector along any loop not passing through the singularities brings the vector back to itself (without any rotation).

1.3.2 Translation surfaces and their strata

Motivation for the study of square-tiled surfaces comes from the theory of *translation surfaces* – arbitrary flat surfaces with conical singularities and trivial holonomy. They can be defined in an elementary way as gluings of arbitrary Euclidean polygons along equal and parallel sides. In a less elementary way, they are equivalent to *holomorphic (Abelian) differentials* on compact Riemann surfaces, which makes their study closely related to the study of the moduli spaces of Riemann surfaces. Some excellent introductory surveys on translation surfaces are [Wri16], [Wri15], [Zor06], [MT02].

The set of translation surfaces with fixed number and angles of the singularities (*singularity profile*) is parametrized by a finite-dimensional orbifold, called a *stratum* of translation surfaces. It turns out that the square-tiled surfaces with the given singularity profile represent the "integer points" of the corresponding stratum for some natural integral affine structure on the latter (given by the so-called *period coordinates*). Because of that the computation of the volume of a stratum for this affine structure (*Masur–Veech volume*) is equivalent to the *asymptotic enumeration* of square-tiled surfaces in this stratum (think of computing the volume of a body in $\mathbb{R}^n$ versus counting the number of integer points in its successively bigger dilations). This approach appears in the paper [Zor02], to which we refer the reader for details.

We now make this statement more precise. For $k = (k_1, \ldots, k_s)$ with $k_i \geq 1$, let $\mathcal{H}(k)$ denote the stratum of translation surfaces with angles of conical singularities equal to $2\pi(k_i + 1)$. The genus of the corresponding surfaces is then uniquely determined by the Gauss-Bonnet formula: $k_1 + \ldots + k_s = 2g - 2$. Let also $\mathcal{ST}(\mathcal{H}(k), N)$ denote the set of square-tiled surfaces in $\mathcal{H}(k)$ with at most N squares and let $d = 2g + s - 1$ be the complex dimension of $\mathcal{H}(k)$. Then

$$\mathrm{Vol}(k) = 2d \cdot \lim_{N \to +\infty} \frac{|\mathcal{ST}(\mathcal{H}(k), N)|}{N^d}. \tag{1.2}$$

The precise knowledge of Masur–Veech volumes is important for the study of a natural ergodic dynamical system on the strata of translation surfaces (*Teichmüller flow*), which has important applications for such natural classes of dynamical systems as billiards in rational polygons or interval exchange transformations.

Geometry and dynamics of translation surfaces is a rich subject which has its origins in the papers [Mas82], [Vee82] of Masur and Veech from the 80's. It has enjoyed a rapid development over the last 20 years, with such keystone results (to name a few) as: equidistribution of geodesics [KMS86]; asymptotic growth of the number of saddle connections and closed geodesics [Vee98], [EM01]; deviations of long geodesics from their asymptotic cycle [Zor97], [Zor99]; properties of the Lyapunov exponents of the Teichmüller flow [For02], [AV07], [EKZ14]; characterisation of orbit closures inside strata of translation surfaces [EM18], [EMM15].

1.3.3 Masur–Veech volumes

The approach to Masur–Veech volumes via counting of square-tiled surfaces appears in the paper of Zorich [Zor02], where he computes the volumes for several strata corresponding to small genus. However, no unified combinatorial approach was proposed. Eskin and Okounkov [EO01], using the representation theory of the symmetric group, developed an algorithm for the computation of the volumes. However, they did not get explicit formulas. Interestingly, explicit generating functions and recursions for the volumes were finally obtained via algebraic geometry in the works of Chen, Möller, Sauvaget and Zagier [Sau18], [CMZ18], [CMSZ20], where the volumes were identified with certain intersection numbers on the strata.

1.3.4 Combinatorial approach to Masur–Veech volumes

A combinatorial approach to the computation of Masur–Veech volumes was recently revived (following the genus zero case in [AEZ14]) by Delecroix, Goujard, Zograf and Zorich [DGZZ21] in the context of *principal strata* of *half-translation surfaces*. We explain here their approach to enumeration of square-tiled surfaces which uses the counting functions for metric ribbon graphs. First we explain the terminology.

Half-translation surfaces are flat surfaces with conical singularities whose holonomy is either trivial or $\mathbb{Z}/2\mathbb{Z}$. They are equivalent to *quadratic* differentials on Riemann surfaces. The corresponding square-tiled surfaces are constructed as before, except the gluing rule is relaxed: one should only glue sides in pairs of type horizontal-horizontal or vertical-vertical. The vertices of the corresponding quadrangulations necessarily have degrees divisible by 2. The total angle around a vertex of degree $2k$ is equal to $2k \cdot \pi/2 = k\pi$. In particular, vertices of degree 2 correspond to non-singular points of the flat metric.

For $k = (k_1, \ldots, k_s)$ with $k_i \geq -1, k_i \neq 0$, denote by $\mathcal{Q}(k)$ the stratum of half translation surfaces with s conical singularities of angles $(k_i + 2)\pi$. The genus of the surface can be recovered from the relation $k_1 + \ldots + k_s = 4g - 4$. A principal stratum is a stratum of the form $\mathcal{Q}(1^{4g-4-n}, -1^n)$. For simplicity of exposition, we will only consider the case $n = 0$.

As before, the volume of the stratum is the (properly normalized) coefficient of the top-degree term of the asymptotics of the number of square-tiled surfaces, when the number of square goes to infinity. In [DGZZ21] the authors give a formula for the volume of each principal stratum as a certain combinatorial sum. Before giving the formula, we introduce the *cylinder decomposition* of square-tiled surfaces.

Cylinder decomposition

Consider a square-tiled surface S with its singular flat metric. Since the number of squares is finite, every horizontal side of every square is either a part of a geodesic joining conical singularities (these can coincide), which we call a *horizontal saddle connection*, or a part of a simple closed geodesic not passing through any singularities. Let G_S be the union of all conical singularities and horizontal saddle connections of S. G_S is a (possibly disconnected) ribbon graph, whose vertices are conical singularities and whose edges are horizontal saddle connections. Moreover it has an integer metric induced from the flat metric of S. The length of any edge is equal to the number of squares incident to it on each side.

Consider the complement $S \setminus G_S$. The closure in S of any connected component of $S \setminus G_S$ carries a non-singular flat metric with geodesic boundary, so, by Gauss-Bonnet theorem, it is a (square-tiled) cylinder. Each cylinder has a positive integer height h_i and circumference L_i. Note that the boundary components of cylinders are glued to the boundary components of the ribbon graphs. In particular, the circumference of a cylinder coincides with the perimeter of the boundary component it is glued to (there are two such boundary components).

This construction shows that any square-tiled surface has a canonical decomposition into square-tiled cylinders, glued together along a collection of integer metric ribbon graphs. To encode the gluing itself, we introduce its *stable graph* Γ.

The vertices of Γ correspond to the metric ribbon graphs of the decomposition. Its edges correspond to the cylinders. An edge joins two vertices (which may coincide) if the corresponding cylinder is glued to the two corresponding ribbon graphs. Each vertex $v \in V(\Gamma)$ is decorated by the genus g_v of the corresponding ribbon graphs.

An automorphism of a stable graph Γ is its graph automorphism which preserves the decorations of the vertices. Denote the set of automorphisms of Γ by $\mathrm{Aut}(\Gamma)$.

Clearly, Γ is connected. Note that the decorations g_v satisfy

$$\sum_{v \in V(\Gamma)} g_v + h_1(\Gamma) = g, \tag{1.3}$$

where $h_1(\Gamma)$ is the first Betti number of Γ, which can be computed as $|E(\Gamma)| - |V(\Gamma)| + 1$. This can be seen for example by a simple application of Euler's formula.

Count of square-tiled surfaces

Let now Γ be an abstract stable graph of genus g, i.e. a connected graph with vertices decorated by non-negative integers g_v satisfying (1.3).

We are now ready to give the formula for the number $|\mathcal{ST}_\Gamma(\mathcal{Q}(1^{4g-4}), N)|$ of square-tiled surfaces $S \in \mathcal{Q}(1^{4g-4})$ with at most N squares and with a fixed (connected) stable graph Γ.

First note that all of the vertices of ribbon graphs in the decomposition of S must have degree 3. Indeed, the vertices correspond to conical singularities. The angle around each of them is equal to 3π. Hence there are exactly three horizontal saddle connections emanating from this singularity.

To construct S, choose for each edge $e \in E(\Gamma)$ a pair of positive integers h_e, L_e, such that $\sum_{e \in E(\Gamma)} h_e L_e \leq N$. For each $e \in E(\Gamma)$ take a square-tiled cylinder of height h_e and circumference L_e. Because of the inequality, the total number of squares is at most N.

Next, for every $v \in V(\Gamma)$ let n_v be the degree of v and let

$$L_v = \{L_e : \ e \text{ is incident to } v\}.$$

For each $v \in V(\Gamma)$ choose an integer metric ribbon graph of genus g_v with n_v boundary components of perimeters L_v. This can clearly be done in $\mathcal{N}_{g_v,n_v}^{[3^{4g_v+2n_v-4}]}(L_v)$ ways.

Finally, glue each cylinder to the pair of corresponding boundary components of ribbon graphs. If the circumference of the cylinder is L_i, there are L_i different ways to perform this gluing. Indeed, one can "twist" the cylinder by a certain number of squares before gluing, and twisting by L_i produces the same square-tiled surface (they differ by a Dehn twist).

Because Γ is connected, the constructed surface S is connected. Since the circumferences of cylinders coincide with the perimeters of the boundary components they are glued to, S is indeed square-tiled. The ribbon graphs are trivalent, so all the conical singularities have cone angles 3π. Moreover, because of (1.3), S is of genus g. Hence $S \in \mathcal{Q}(1^{4g-4})$.

Taking the symmetries of Γ into account we finally obtain

$$|\mathcal{ST}_\Gamma(\mathcal{Q}(1^{4g-4}), N)| = \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\sum_e h_e L_e \leq N} \prod_{e \in E(\Gamma)} L_e \prod_{v \in V(\Gamma)} \mathcal{N}_{g_v,n_v}^{[3^{4g_v+2n_v-4}]}(L_v). \tag{1.4}$$

Computing the volume

Condition (1.3) implies that only a finite number of stable graphs appear as stable graphs of square-tiled surfaces in $\mathcal{Q}(1^{4g-4})$. Indeed, rewrite (1.3) as $\sum_{v\in V(\Gamma)}(g_v + \deg_\Gamma(v)/2) = |V(\Gamma)| + g - 1$. If $|V(\Gamma)|$ is big enough, at least one of the terms in the sum is less then or equal to 1. But this is only possible if $g_v = 0$ and $\deg_\Gamma(v) \in \{1, 2\}$. However, there are no trivalent ribbon graphs in genus 0 with 1 or 2 boundary components, a contradiction.

Hence the volume of $\mathcal{Q}(1^{4g-4})$ breaks down into a finite number of contributions coming from square-tiled surfaces with a fixed stable graph. The contribution corresponding to Γ is the coefficient of the top-degree term of the $N \to \infty$ asymptotics of (1.4).

To compute this asymptotics, one can use the following elementary lemma [AEZ14, Lemma 3.7].

Lemma 1.11. *Let $n \geq 1$ and $s_1, \ldots, s_n \in \mathbb{Z}_{>0}$. Then, as $N \to \infty$,*

$$\sum_{\substack{\sum_{i=1}^n h_i L_i \leq N \\ h_i, L_i \in \mathbb{Z}_{>0}}} L_1^{s_1} \cdots L_n^{s_n} \sim \frac{N^{s+n}}{(s+n)!} \cdot \prod_{i=1}^n (s_i! \cdot \zeta(s_i + 1)),$$

where $s = s_1 + \ldots + s_n$ and ζ is the Riemann zeta function.

By linearity, Lemma 1.11 allows to compute the asymptotics of any such sum where the function being summed is a polynomial in the variables $L_1, \ldots, L_n$ divisible by $L_1 \cdots L_n$. Only the terms of top degree contribute to the total asymptotics.

Hence one can replace in (1.4) the counting functions $\mathcal{N}_{g_v,n_v}^{[3^{4g_v+2n_v-4}]}(L_v)$ by their top-degree terms $N_{g_v,n_v}(L_v)$ (Theorem 1.8), and apply Lemma 1.11.

Summing over all stable graphs Γ we obtain the volume of $\mathcal{Q}(1^{4g-4})$.

Bibliographic remarks

An intersection theory interpretation of the volumes of the principal strata of quadratic differentials does exist [CMS23]. Nevertheless, the combinatorial approach described above has certain advantages.

Recall that the coefficients of $N_{g_v,n_v}(L_v)$ are the intersections of ψ-classes on the moduli spaces of curves. It is easy to see that the expression for the volume of $\mathcal{Q}(1^{4g-4})$ that we have just constructed is in fact a polynomial in the intersections of ψ-classes. In a recent paper [Agg21], Aggarwal computed the large genus asymptotics of the intersections of ψ-classes. Using the formula of [DGZZ21] constructed above, he then was able to deduce the large genus asymptotics of the volumes of the principal strata.

Another interesting feature of the combinatorial approach is as follows. Let Vol_Γ be the contribution to the volume of a stratum of the square-tiled surfaces with stable graph Γ. Let also Vol be the total volume. The ratio $\mathrm{Vol}_\Gamma / \mathrm{Vol}$ can be interpreted as the probability of a random square-tiled surface (of a very big size) to have stable graph Γ. Similar ideas were developed in [DGZZ22] to study the geometric properties of random square-tiled surfaces such as the number and the heights of the cylinders. Unexpectedly, all these statistics of square-tiled surfaces turned out [DGZZ22] to be equal to the statistics of random geodesic multicurves on hyperbolic surfaces such as the topological type, the number and the weights of the components. This inspired several further papers studying the length statistics of the components [Liu22], [DL22].

1.3.5 Motivation for the thesis

The initial motivation for this thesis was to extend the methods of [DGZZ21] to strata of translation surfaces. As we will see (Proposition 3.13), this amounts to the study of counting functions for face-bipartite metric ribbon graphs.

The crucial point in the construction of [DGZZ21] is that the top-degree terms of the counting functions for trivalent metric ribbon graphs are in fact polynomials with explicit coefficients. This naturally leads us to the question: are there families of face-bipartite ribbon graphs with such polynomiality property? We have been able to answer positively to this question in several particular cases, see next section.

1.4 Contributions of the thesis

Chapter 2: preliminaries

We prove several elementary properties of weight functions on ribbon graphs. These are arbitrary real functions on the set of edges of the ribbon graph and are generalizations of metrics. We distinguish the cases of face-bipartite and non-face-bipartite ribbon graphs. Even though they are similar, they present several subtle differences.

Using these properties we show that the counting functions of face-bipartite ribbon graphs are in general piecewise polynomial, and the ones of non-face-bipartite ribbon graphs are piecewise quasi-polynomial. We also specify the regions of (quasi-)polynomiality. The proofs are based on a result from the theory of enumeration of integer points in polyhedra.

Even though these polynomiality properties are known to hold in some particular cases, they were not stated in this generality before. Moreover, our applications require the knowledge of the regions of (quasi-)polynomiality. This motivated us to include these results here.

Chapter 3: one-vertex graphs

In this chapter we consider the counting functions of face-bicolored metric ribbon graphs with one vertex. We show that their top-degree terms are polynomial outside of a finite number of hyperplanes ("walls"), and that a similar result is true for the top-degree terms of their restrictions to certain linear subspaces of interest. Moreover, we give explicit formulas for these top-degree terms. The proofs are purely combinatorial. The proof in genus 0 relies on a simple construction involving flips of edges in plane trees. The proof in positive genus relies on a result from the theory of combinatorial maps: the Chapuy-Féray-Fusy bijection between ribbon graphs with one face and plane trees decorated with permutations.

Next, using the obtained top-degree terms of the counting functions, we compute the generating series of contributions of n-cylinder square-tiled surfaces to the Masur-Veech volumes of minimal strata $\mathcal{H}(2g-2)$ of Abelian differentials. This result is a refinement of a previous result of Sauvaget [Sau18] obtained using intersection theory, which gives the generating series of the total volumes.

The material in this chapter is drawn from my paper [Yak23].

Chapter 4: spin parity of one-vertex graphs

We present a conditional theorem which gives the generating series of the differences of contributions of n-cylinder square-tiled surfaces to the spin connected components of the minimal strata $\mathcal{H}(2g-2)$ of Abelian differentials. This series is a refinement of the generating series for the total volume differences obtain previously by Chen, Möller, Sauvaget and Zagier [CMSZ20] using intersection theory.

The theorem is conditional because it relies on a yet unproven property of certain counting functions. More precisely, we conjecture the existence of an invariant of face-bipartite ribbon graphs with one vertex called combinatorial spin parity. It is a combinatorial analog of the topological invariant used to distinguish the connected components of the minimal stratum. The unproven property is that the counting functions for the families of ribbon graphs with even/odd combinatorial spin parity have polynomial top-degree terms which are equal (except in one base case).

Chapter 5: prefix-postfix sequences of trees

We introduce an invariant of rooted vertex-bicolored plane trees with labeled vertices called the prefix-postfix sequence.

We prove that for any (generic) vertex perimeters and any (cyclic equivalence class of) prefix-postfix sequence, there exists a unique metric tree with these parameters. This can be seen as a polynomiality property for the counting function of the family of trees with given (class of) prefix-postfix sequence. Namely, this function is a constant equal to 1. This result is a crucial part of the proof in Chapter 6 of the polynomiality of the weighted counting functions for plane many-vertex face-bipartite ribbon graphs. Its proof is inspired by the proof of polynomiality in genus 0 from Chapter 3: we show that the edges of rooted plane trees can be flipped in such a way as to preserve the (class of the) prefix-postfix sequence.

Next, we show that each family of trees with a fixed (class of) prefix-postfix sequence gives rise to a triangulation of the product of two simplices. We give a recursive construction for such triangulations and give a precise conjecture on the number of distinct triangulations obtained in this way. The triangulations of this polytope are interesting from the point of view of the theory of polytopes. Our triangulations seem to have not been considered before.

Finally, in Chapter 4 we prove that for trees with equal number of black and white vertices, the parity of a permutation canonically associated to the prefix-postfix sequence coincides with the spin parity of this tree (Lemma

4.11).

Chapter 6: many-vertex graphs

The top-degree terms of the counting functions for many-vertex face-bipartite ribbon graphs are not polynomial in general. However, we conjecture that, if the contribution of each ribbon graph is weighted by the count of certain spanning trees, the top-degree term does become polynomial outside of a finite number of hyperplanes. Moreover, this polynomial only depends on the number of vertices and not on their degrees.

We prove the conjecture for plane ribbon graphs. The two main ingredients of the proof are the result from Chapter 5 about the count of metric trees with given vertex perimeters and given class of prefix-postfix sequence, and the bijection due to Bernardi between plane ribbon graphs equipped with a spanning tree and pairs of plane trees.

Chapter 7: metric ribbon graphs with odd vertex degrees

We study the counting functions for ribbon graphs with odd vertex degrees. Their top-degree terms are known (due to the work of Kontsevich) to be polynomial outside of a finite number of hyperplanes. We generalize this result by showing that the top-degree terms of their restrictions to certain linear subspaces of interest are also polynomial, and we show how they can be computed recursively. The proof relies on the precise study of degenerations of metric ribbon graphs and boils down to the proof of a formula for the coefficients counting certain particular degenerations. This formula is proved using a Prüfer-code-style bijection for degenerated ribbon graphs. This proof is my contribution to a joint project with Duryev and Goujard (paper in preparation), where the combinatorial formulas of [DGZZ21] for the volumes of principal strata of quadratic differentials are generalized to strata of quadratic differentials with odd degrees of zeros.

Chapter 2

Preliminaries

In our study of metrics on ribbon graphs we will need to assign to the edges of ribbon graphs possibly non-positive numbers. We call such assignments the *weight functions*. In this chapter we prove several general elementary statements about the properties of weight functions, distinguishing the cases of bipartite (section 2.1.1) and non-bipartite (section 2.1.2) ribbon graphs.

Using these properties, we prove that the functions counting metrics on a ribbon graph G with given *vertex* perimeters are piecewise polynomial if G is bipartite and piecewise quasi-polynomial of G is non-bipartite (Propositions 2.16 and 2.17). We also specify exactly the regions of (quasi-)polynomiality. The latter are the open cells of a certain polyhedral subdivision of the space of vertex perimeters. This subdivision is generated by a certain family of hyperplanes, which we call the "walls" (section 2.2.1). In section 2.2.2 we give a strategy of proof, while the formal proofs are postponed to sections 2.3.1 and 2.3.2).

Finally, in sections 2.2.4 and 2.2.5, we explain a general strategy for the computation of the top-degree terms of the counting functions on the walls and their intersections, by counting degenerations of ribbon graphs. It will be applied in different contexts in later chapters.

2.1 Weight functions on ribbon graphs

Definition 2.1 (Weight functions)**.** A *weight function* on a ribbon graph G is a function $w : E(G) \to \mathbb{R}$. We denote $w(e) = w_e$ and call it the *weight* of the edge $e \in E(G)$. A weight function is *non-negative* (*positive*, *integral*) if all the weights $w_e, e \in E(G)$ are non-negative (positive, integral respectively).

Note that metrics on G are exactly the positive weight functions on G.

Definition 2.2 (Vertex perimeters)**.** The *perimeter of a vertex* $v \in V(G)$ for a weight function w on G is the sum of weights w_e for all edges e incident to v. If e is a loop based at v, w_e is counted twice in the sum.

The space $\mathbb{R}^{E(G)}$ of all possible weight functions on G is naturally a vector space. Denote by $\mathrm{vp}_G : \mathbb{R}^{E(G)} \to \mathbb{R}^{V(G)}$ the linear map that sends a weight function to the perimeters it gives to the corresponding vertices.

If $G \in \mathcal{RG}^*_{g,n}$ is a ribbon graph with n vertices labeled from 1 to n, we will identify $\mathbb{R}^{V(G)}$ with $\mathbb{R}^n$ and denote by $L_1, \ldots, L_n$ the standard coordinates on the latter.

If $G \in \mathcal{RG}^*_{g,(k,l)}$ is vertex-bicolored ribbon graph with k black vertices labeled from 1 to k and l white vertices labeled from 1 to l, we will identify $\mathbb{R}^{V(G)}$ with $\mathbb{R}^k \times \mathbb{R}^l$ and denote by $L_1, \ldots, L_n; L'_1, \ldots, L'_l$ the standard coordinates on the latter.

We now prove several elementary properties of the weight functions. We distinguish the cases of bipartite and non-bipartite ribbon graphs, because they have slight differences. The results in both cases are similar, and we summarize the differences in Table 2.1.

	bipartite	non-bipartite
$\mathrm{Im}(\mathrm{vp}_G)$	$H_{k,l}$	$\mathbb{R}^n$
Edges whose weight is a function of vertex perimeters	bridges $B(G)$	static edges $S(G)$
Graphs with unique weight function, given vertex perimeters	trees	one odd cycle (OOC) graphs
$\dim \mathrm{vp}_G^{-1}(\cdot)$	$\|E(G)\| - \|V(G)\| + 1$	$\|E(G)\| - \|V(G)\|$
$\mathrm{vp}_G^{-1}(\cdot) \cap \mathbb{Z}^{E(G)}$ is a full rank lattice iff	$(L, L') \in (\mathbb{Z}^k \times \mathbb{Z}^l) \cap H_{k,l}$	$L \in \mathbb{Z}^n, \sum L_i$ is even

Table 2.1: Weight functions on bipartite vs non-bipartite ribbon graphs.

2.1.1 Bipartite ribbon graphs

Denote by $H_{k,l}$ the subspace of $\mathbb{R}^k \times \mathbb{R}^l$ defined by the equation

$$L_1 + \ldots + L_k = L'_1 + \ldots + L'_l.$$

Lemma 2.3. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph. If w is a weight function on G with $\mathrm{vp}_G(w) = (L, L')$, then $L_1 + \ldots + L_k = L'_1 + \ldots + L'_l$.*

In other terms, $\mathrm{Im}(\mathrm{vp}_G) \subset H_{k,l}$.

Proof. Every edge contributes its weight to the perimeters of both its black and white extremities. Hence both the sum of the perimeters of black vertices of G and the sum of the perimeters of its white vertices are equal to $\sum_{e\in E(G)} w_e$. $\square$

Let $B(G) \subset E(G)$ be the set of *bridges* of G, i.e. edges whose deletion disconnects G. The following lemma states that the length of a bridge is uniquely determined by the vertex perimeters.

Lemma 2.4 (Bridge weight)**.** *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph and let w be a weight function on G with $\mathrm{vp}_G(w) = (L, L')$. Let also $e \in B(G)$ be a bridge. Then*

$$w_e = \sum_{i\in I} L_i - \sum_{j\in J} L'_j = \sum_{j\in J^c} L'_j - \sum_{i\in I^c} L_i, \tag{2.1}$$

where $I \subset \{1, \dots, k\}$ and $J \subset \{1, \dots, l\}$ are the labels of black and white vertices in the connected component of $G - e$ containing the black extremity of e.

Proof. Denote by $(G - e)_b$ the connected component of $G - e$ containing the black extremity of e. Again we note that every edge contributes its weight to the perimeters of both its black and white extremities. Hence the sum of edge weights in $(G - e)_b$ is $\sum_{j\in J} L'_j$ (total contribution to white vertices of $(G - e)_b$) and at the same time $\sum_{i\in I} L_i - w_e$ (total contribution to black vertices of $(G - e)_b$). The first equality follows. The second equality follows from Lemma 2.3. $\square$

For a bridge $e \in B(G)$ we denote by $f_e(L, L')$ the linear function on $H_{k,l}$ given by (2.1). Hence $f_e(L, L')$ is the weight of e as a function of vertex perimeters. We do not specify the dependency on G in the notation f_e as it will always be clear from the context.

Lemma 2.5 (Weight functions on trees)**.** *Let $G \in \mathcal{E}^*_{0,k,l}$ be a tree. Then the linear map $\mathrm{vp}_G : \mathbb{R}^{E(G)} \to \mathbb{R}^k \times \mathbb{R}^l$ induces an isomorphism between $\mathbb{R}^{E(G)}$ and $H_{k,l}$. Moreover, $w \in \mathbb{Z}^{E(G)}$ if and only if $\mathrm{vp}_G(w) \in \mathbb{Z}^k \times \mathbb{Z}^l$.*

Proof. We first show the vp_G is injective. Suppose w is such that $\mathrm{vp}_G(w) = (L, L')$. Since in a tree all edges are bridges, by Lemma 2.4 $w_e = f_e(L, L')$ for every $e \in E(G)$. Hence such w is necessarily unique. We now show that $\mathrm{Im}(\mathrm{vp}_G) = H_{k,l}$. By Lemma 2.3 $\mathrm{Im}(\mathrm{vp}_G) \subset H_{k,l}$. In the other direction, let $(L, L') \in H_{k,l}$. Construct a weight function w by setting $w_e = f_e(L, L')$ for every $e \in E(G)$. The proof that $\mathrm{vp}_G(w) = (L, L')$ is then a simple computation.

The integrality of w implies $\mathrm{vp}_G(w) \in \mathbb{Z}^k \times \mathbb{Z}^l$ by definition of vp_G. Opposite implication holds because linear forms in $f_e(L, L')$ have integer coefficients. □

Lemma 2.6. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph. Then $\mathrm{Im}(\mathrm{vp}_G) = H_{k,l}$. Moreover, for every $(L, L') \in H_{k,l}$, $\mathrm{vp}_G^{-1}(L, L')$ is an affine subspace of $\mathbb{R}^{E(G)}$ of dimension $|E(G)| - |V(G)| + 1$.*

Proof. Again, $\mathrm{Im}(\mathrm{vp}_G) \subset H_{k,l}$ by Lemma 2.3. To prove the opposite inclusion, fix $(L, L') \in H_{k,l}$. Choose a spanning tree T of G. Set $w_e = 0$ for $e \in E(G) \setminus E(T)$. By Lemma 2.5 there exists a weight function $w' \in \mathbb{R}^{E(T)}$ such that $\mathrm{vp}_T(w') = (L, L')$. Set $w_e = w'_e$ for $e \in E(T)$. Thus constructed weight function $w \in \mathbb{R}^{E(G)}$ satisfies $\mathrm{vp}_G(w) = (L, L')$.

The dimension of $\mathrm{vp}_G^{-1}(L, L')$ is $\dim \mathbb{R}^{E(G)} - \dim H_{k,l} = |E(G)| - (k + l - 1) = |E(G)| - |V(G)| + 1$. □

Lemma 2.7. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph and let $(L, L') \in H_{k,l}$. Regard the coordinates w_e as linear functions on the affine subspace $\mathrm{vp}_G^{-1}(L, L')$. Then for all $e \in B(G)$, w_e is constant with value $f_e(L, L')$. All other functions $w_e, e \in E(G) \setminus B(G)$ are non-constant.*

Proof. The first claim follows from Lemma 2.4. Let now $e \in E(G) \setminus B(G)$. Since e is not a bridge, there exists a cycle in G containing e. Since G is bipartite, this cycle has even length. One can now change the value of w_e while staying inside $\mathrm{vp}_G^{-1}(L, L')$ by alternately adding and subtracting some $t \in \mathbb{R}$ to/from the weights of consecutive edges of this cycle. So w_e is indeed non-constant on $\mathrm{vp}_G^{-1}(L, L')$. □

Lemma 2.8. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph and $(L, L') \in H_{k,l} \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$. Then $\mathrm{vp}_G^{-1}(L, L') \cap \mathbb{Z}^{E(G)}$ is a lattice of full rank in $\mathrm{vp}_G^{-1}(L, L')$.*

Proof. As in the proof of Lemma 2.6, choose a spanning tree T of G. There is a weight function w' on T such that $\mathrm{vp}_T(w') = (L, L')$. By Lemma 2.5 it is integral. Extend w' to a weight function w on G by setting $w(e) = 0$ for $e \in E(G) \setminus E(T)$. We thus have $\mathrm{vp}_G^{-1}(L, L') \cap \mathbb{Z}^{E(G)} = w + \mathrm{vp}_G^{-1}(0, 0) \cap \mathbb{Z}^{E(G)}$. We prove that $\mathrm{vp}_G^{-1}(0, 0) \cap \mathbb{Z}^{E(G)}$ is a lattice of rank $|E(G)| - |V(G)| + 1$ by providing a basis of size $|E(G)| - |V(G)| + 1$.

For each $e \in E(G) \setminus E(T)$ let γ_e be a closed path which is a concatenation of e with the unique path in T joining the endpoints of e. Since G is bipartite, γ_e has even length. Starting from a zero weight function, modify the weight of each edge along the path γ_e by $+1$ and -1 alternately. This gives an integral weight function w^e such that $\mathrm{vp}_G(w^e) = (0, 0)$, $w^e(e) = 1$, $w^e(e') = 0$ for all $e' \in E(G) \setminus E(T)$, $e' \neq e$.

The weight functions $w_e,\ e \in E(G) \setminus E(T)$ form a basis of $\mathrm{vp}_G^{-1}(0,0) \cap \mathbb{Z}^{E(G)}$. $\square$

2.1.2 Non-bipartite ribbon graphs

Recall that a graph is bipartite if and only if all of its cycles have even length. Equivalently, a graph is non-bipartite if and only if it contains a cycle of odd length.

Definition 2.9 (Static edges)**.** Let G be a non-bipartite ribbon graph. An edge $e \in E(G)$ is called *static* if at least one connected component of $G - e$ is bipartite. The set of static edges of a non-bipartite ribbon graph G is denoted by $S(G)$.

Note that if a static edge e is a bridge, then exactly one connected component of $G - e$ is bipartite (because otherwise G would be bipartite).

If e is not a bridge, than $G-e$ is connected and bipartite, and e is incident to two vertices from the same part of $G-e$ (again, because otherwise G would be bipartite).

Lemma 2.10 (Weight of a static edge)**.** *Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph and let w be a weight function on G with $\mathrm{vp}_G(w) = L$. Let also $e \in S(G)$ be a static edge.*

- *If e is a bridge, let G' be the connected component of $G - e$ which is bipartite. Color the vertices of G' in black and white in such a way that e is adjacent to a black vertex. Let $I, J \subset \{1, \dots, n\}$ be the labels of black and white vertices in G'. Then*

$$w(e) = \sum_{i \in I} L_i - \sum_{j \in J} L_j. \tag{2.2}$$

- *If e is not a bridge, $G - e$ is connected and bipartite. Color its vertices in black and white in such a way that e is adjacent to two black vertices. Let $I, J \subset \{1, \dots, n\}$ be the labels of black and white vertices in $G - e$. Then $I \cup J = \{1, \dots, n\}$ and*

$$w(e) = \frac{1}{2} \cdot \left(\sum_{i \in I} L_i - \sum_{j \in J} L_j \right). \tag{2.3}$$

Proof. Compute the sum of all edge lengths of the bipartite component of $G - e$ in two ways. On the one hand it is $\sum_{j \in J} L_j$. On the other hand, it is $\sum_{i \in I} L_i - w(e)$ when e is a bridge, and $\sum_{i \in I} L_i - 2w(e)$ when it is not. $\square$

For a static edge $e \in S(G)$ we denote by $f_e(L)$ the linear function on $\mathbb{R}^n$ giving the weight of e as a function of vertex perimeters, (2.2) or (2.3). Again, we do not specify the dependency on G in the notation f_e as it will always be clear from the context.

Definition 2.11. A *one odd cycle (OOC) graph* is a connected graph with exactly one simple cycle of odd length.

An OOC graph is simply an odd cycle with trees attached to its vertices. Thus an OOC graph on n vertices has exactly n edges. Any OOC graph is clearly non-bipartite.

The following lemma states that for any choice of vertex perimeters, there exists a unique weight function on an OOC ribbon graph with these vertex perimeters.

Lemma 2.12 (Weight functions on OOC ribbon graphs)**.** *Let G be an OOC ribbon graph with n vertices. Then the linear map $\mathrm{vp}_G : \mathbb{R}^{E(G)} \to \mathbb{R}^n$ is an isomorphism. Moreover, $w \in \mathbb{Z}^{E(G)}$ if and only if $\mathrm{vp}_G(w)$ is in the sublattice of $\mathbb{Z}^n$ defined by $L_1 + \ldots + L_n = 0 \pmod 2$.*

Proof. Note that every edge of G is static. Indeed, deletion of any edge produces either a tree, or an OOC graph and a tree. Trees are bipartite.

Hence, by Lemma 2.10, given the vertex perimeters L, the weight of each edge is uniquely determined. So vp_G is injective. For the surjectivity, note that assigning to each edge the weight given by Lemma 2.10 produces the necessary weight function.

$L_1 + \ldots + L_n$ is twice the sum of weights of all edges. So if w is integral, this sum is necessarily even. Conversely, if $L_1 + \ldots + L_n = 0 \pmod 2$, the weights of all edges are integral by Lemma 2.10. $\square$

Lemma 2.13. *Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph. Then $\mathrm{Im}(\mathrm{vp}_G) = \mathbb{R}^n$. Moreover, for every $L \in \mathbb{R}^n$, $\mathrm{vp}_G^{-1}(L)$ is an affine subspace of $\mathbb{R}^{E(G)}$ of dimension $|E(G)| - |V(G)|$.*

Proof. Choose a spanning tree T of G. Complete it to an OOC graph T' by adding one edge (such an edge exists since G is non-bipartite). By Lemma 2.12 there is a weight function w' on T' such that $\mathrm{vp}_{T'}(w') = L$. Extend w' to a weight function w on G by setting $w(e) = 0$ if $e \notin E(T')$. Then $\mathrm{vp}_G(w) = L$, and so vp_G is surjective. Consequently, for any L, the dimension of $\mathrm{vp}_G^{-1}(L)$ is $\dim\ker \mathrm{vp}_G = |E(G)| - n = |E(G)| - |V(G)|$. $\square$

Lemma 2.14. *Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph and let $L \in \mathbb{R}^n$. Regard the coordinates w_e of $\mathbb{R}^{E(G)}$ as linear functions on $\mathrm{vp}_G^{-1}(L)$. Then for all $e \in S(G)$, w_e is constant with value $f_e(L)$. All other functions $w_e, e \in E(G) \setminus S(G)$ are non-constant.*

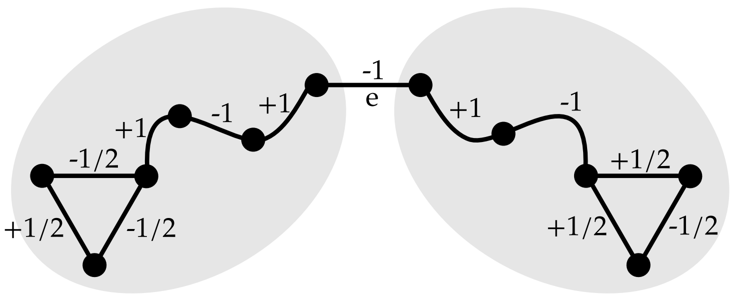


Figure 2.1: A way to change the weight of a non-static edge $e \in E(G) \setminus S(G)$ without changing the vertex perimeters.

Proof. First claim follows from Lemma 2.10. Let now $e \in E(G) \setminus S(G)$.

Let v_1 and v_2 be the endpoints of e (these might coincide). Since all connected components of $G-e$ are non-bipartite, there are odd cycles C_1 and C_2 in the connected components of $G - e$ containing v_1 and v_2, respectively. Let γ_1 and γ_2 be paths in $G - e$ connecting v_1 to C_1 and v_2 to C_2. Let γ be the (non-simple) path in G which is a concatenation (in this order) of γ_1, C_1, γ_1 in reverse, e, γ_2, C_2, γ_2 in reverse, e.

γ is a closed path of even length. Change the weights of successively visited edges of γ by $+1/2$ and $-1/2$ alternately. This changes the weight of e by -1, while preserving all vertex perimeters (see Figure 2.1 for an example). Thus w_e is not constant on $\mathrm{vp}_G^{-1}(L)$. □

Lemma 2.15. *Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph and $L \in \mathbb{Z}^n$. If $L_1 + \ldots + L_n = 0 \pmod 2$, then $\mathrm{vp}_G^{-1}(L) \cap \mathbb{Z}^{E(G)}$ is a lattice of full rank in $\mathrm{vp}_G^{-1}(L)$. Otherwise, $\mathrm{vp}_G^{-1}(L) \cap \mathbb{Z}^{E(G)}$ is empty.*

Proof. If $L_1 + \ldots + L_n$ is odd, there is clearly no integer weight function on G with vertex perimeters L, and so $\mathrm{vp}_G^{-1}(L) \cap \mathbb{Z}^{E(G)}$ is empty.

Suppose $L_1 + \ldots + L_n$ is even. As in the proof of Lemma 2.13, choose a spanning tree T of G and complete it to an OOC graph T' by adding one edge. There is a weight function w' on T' such that $\mathrm{vp}_{T'}(w') = L$. By Lemma 2.12 it is integral. Extend w' to a weight function w on G by setting $w(e) = 0$ for $e \in E(G) \setminus E(T')$. We thus have $\mathrm{vp}_G^{-1}(L) \cap \mathbb{Z}^{E(G)} = w + \mathrm{vp}_G^{-1}(0) \cap \mathbb{Z}^{E(G)}$. We prove that $\mathrm{vp}_G^{-1}(0) \cap \mathbb{Z}^{E(G)}$ is a lattice of rank $|E(G)| - n$ by providing a basis of size $|E(G)| - n$.

For each $e \in E(G) \setminus E(T')$ construct a closed path γ_e as follows. Let v_1, v_2 be the endpoints of e, let γ_1 and γ_2 be the paths in T' connecting v_1 and v_2 to the unique odd cycle C of T'. Then γ_e is a concatenation of γ_1, C, γ_1 in reverse, e, γ_2, C, γ_2 in reverse, e. γ_e is a path of even length. Starting from a zero weight function, modify the weight of each edge along the path γ_e by

$+1/2$ and $-1/2$ alternately. This gives an integral weight function w^e such that $\mathrm{vp}_G(w^e) = 0$, $w^e(e) = 1$, $w^e(e') = 0$ for all $e' \in E(G) \setminus E(T')$, $e' \neq e$.

The weight functions w_e, $e \in E(G) \setminus E(T')$ form a basis of $\mathrm{vp}_G^{-1}(0) \cap \mathbb{Z}^{E(G)}$. □

2.2 Properties of counting functions

2.2.1 Walls / notations

We introduce here a family of hyperplanes ("walls") in the space of vertex perimeters, which will determine the regions of (quasi-)polynomiality of the counting functions. We also introduce some notations the will be used throughout the thesis. As before, we have two parallel cases, for bipartite and non-bipartite ribbon graphs.

Bipartite case

For integers $k, l \geq 1$:

- let $H_{k,l}$ be the hyperplane in $\mathbb{R}^k \times \mathbb{R}^l$ given by

$$L_1 + \ldots + L_k = L'_1 + \ldots + L'_l; \tag{2.4}$$

- let $H^+_{k,l}$ be the cone $H_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$;
- let $\mathcal{W}_{k,l}$ denote the set of hyperplanes of $H_{k,l}$ (the *walls*) of the form

$$\sum_{i \in I} L_i = \sum_{j \in J} L'_j,$$

  where $I \subset \{1, \ldots, k\}$, $J \subset \{1, \ldots, l\}$, $(I, J) \neq (\varnothing, \varnothing)$, $(I^c, J^c) \neq (\varnothing, \varnothing)$;
- let $\overline{\mathcal{W}_{k,l}}$ be the set of linear subspaces of $H_{k,l}$ which are intersections of several hyperplanes from $\mathcal{W}_{k,l}$ (empty intersection corresponds to $H_{k,l}$ itself);
- for any $W \in \overline{\mathcal{W}_{k,l}}$ let $W^\circ = W - \bigcup_{V \in \overline{\mathcal{W}_{k,l}}, V \subsetneq W} V$, i.e. W minus the subspaces from $\overline{\mathcal{W}_{k,l}}$ of smaller dimension included in W. Note that the sets W° for $W \in \overline{\mathcal{W}_{k,l}}$ form a partition of $H_{k,l}$.
- let $\mathcal{L}_{k,l}$ be the set of linear functions on $H_{k,l}$ of the form

$$\sum_{i \in I} L_i - \sum_{j \in J} L'_j,$$

where $I \subset \{1,\ldots,k\}$, $J \subset \{1,\ldots,l\}$, $(I,J) \neq (\varnothing,\varnothing)$, $(I^c,J^c) \neq (\varnothing,\varnothing)$; note that the hyperplanes in $\mathcal{W}_{k,l}$ are exactly the kernels of functions from $\mathcal{L}_{k,l}$.

Non-bipartite case

For $n \geq 1$:

- let $\mathcal{W}_n$ denote the set of hyperplanes of $\mathbb{R}^n$ (the *walls*) of the form

$$\sum_{i\in I} L_i = \sum_{j\in J} L_j,$$

  where $I, J \subset \{1,\ldots,n\}$ are disjoint and $(I,J) \neq (\varnothing,\varnothing)$;

- let $\overline{\mathcal{W}_n}$ be the set of linear subspaces of $\mathbb{R}^n$ which are intersections of several hyperplanes from $\mathcal{W}_n$ (empty intersection corresponds to $\mathbb{R}^n$ itself);

- for any $W \in \overline{\mathcal{W}_n}$ let $W^\circ = W - \bigcup_{V\in\overline{\mathcal{W}_n}, V\subsetneq W} V$, i.e. W minus the subspaces from $\overline{\mathcal{W}_n}$ of smaller dimension included in W. Note that the sets W° for $W \in \overline{\mathcal{W}_n}$ form a partition of $\mathbb{R}^n$.

- let $\mathcal{L}_n$ be the set of linear functions on $\mathbb{R}^n$ of the form

$$\sum_{i\in I} L_i - \sum_{j\in J} L_j,$$

  where $I, J \subset \{1,\ldots,n\}$ are disjoint and $(I,J) \neq (\varnothing,\varnothing)$; note that the hyperplanes in $\mathcal{W}_n$ are exactly the kernels of functions from $\mathcal{L}_n$.

- let $\frac{1}{2}\mathcal{L}_n = \{\frac{1}{2}f, f \in \mathcal{L}_n\}$.

2.2.2 Piecewise (quasi-)polynomiality

For $n \geq 1$ denote by $\mathcal{PS}_n$ the polyhedral subdivision of $\mathbb{R}^n$ generated by the union of the walls $\bigcup_{W\in\mathcal{W}_n} W$. The open cells of this polyhedral subdivision are the connected components of W° for all subspaces $W \in \overline{\mathcal{W}_n}$. Note also that each open cell is a cone.

For example, for $n = 2$, there are 4 walls in $\mathcal{W}_2$, given by the equations $L_1 = 0$, $L_2 = 0$, $L_1 + L_2 = 0$, $L_1 = L_2$. They generate the polyhedral subdivision $\mathcal{PS}_2$ of $\mathbb{R}^2$ with 8 open cells of dimension 2, 8 open cells of dimension 1, and 1 open cell of dimension 0 (Figure 2.2, left).

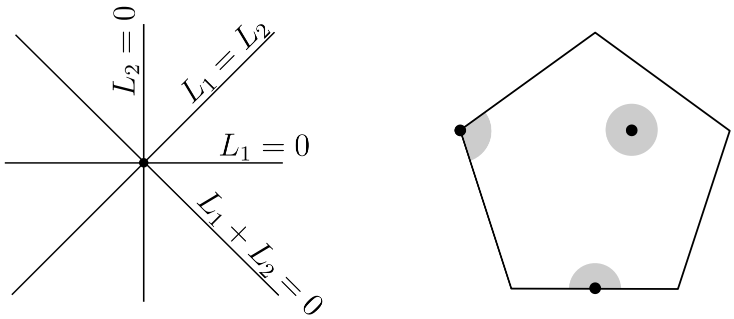


Figure 2.2: Left: polyhedral subdivision $\mathcal{PS}_2$ of $\mathbb{R}^2$. Right: cones of feasible directions to a polygon in $\mathbb{R}^2$ at three points.

Similarly, for $k, l \geq 1$, denote by $\mathcal{PS}_{k,l}$ the polyhedral subdivision of $H_{k,l}$ generated by the union of the walls $\bigcup_{W \in \mathcal{W}_{k,l}} W$.

Recall the duality Convention 1.7.

Proposition 2.16. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph and let C be an open cell in $\mathcal{PS}_{k,l}$. Then for $(L, L') \in C \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$, the function $\mathcal{P}_G(L, L')$ is either identically zero or is a polynomial of degree $|E(G)| - |V(G)| + 1$.*

Proposition 2.17. *Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph and let C be an open cell in $\mathcal{PS}_n$. Then for $L \in C$ and in a fixed coset of $2\mathbb{Z}^n \subset \mathbb{Z}^n$, the function $\mathcal{N}_G(L)$ is either identically zero or is a polynomial of degree $|E(G)| - |V(G)|$.*

Functions on $\mathbb{Z}^n$ which are polynomial on the cosets of some sublattice of $\mathbb{Z}^n$ of finite index are commonly referred to as *quasi-polynomials*. Thus the functions $\mathcal{N}_G$ are *piecewise quasi-polynomials*.

Propositions 2.16 and 2.17 obviously imply that the counting functions of the corresponding families of ribbon graphs are also polynomial or quasi-polynomial on each open cell of the corresponding polyhedral subdivision.

We explain the strategy of the proof of these statements in the next section. It boils down to the problem of enumeration of integer points in certain polytopes (*polytopes of metrics*). The proofs themselves are postponed to sections 2.3.1 and 2.3.2.

Results similar to Propositions 2.16 and 2.17 have been obtained before, and the idea of their proof (counting integer points in polytopes) is not new. For the non-bipartite case, see for example [Nor13, section 3]. The piecewise polynomiality results analogous to Proposition 2.16 (with the same set of walls) are known to hold for *double Hurwitz numbers*, which count ramified covers of the sphere with two branch points with given branching behavior and an appropriate number of simple branch points. See [GJV05], [SSV08], [CJM11].

Nevertheless, we present here the detailed proofs of these Propositions, because our applications require: (a) the precise knowledge of the regions of (quasi-)polynomiality, including the lower-dimensional cells; (b) the precise relation between the discontinuities of the counting function and the structure of the corresponding ribbon graph.

2.2.3 Polytopes of metrics

In what follows we will use some terminology coming from the theory of polyhedra. We refer the reader to [Bar08] for details.

Bipartite graphs

Suppose $G \in \mathcal{RG}^*_{g,(k,l)}$ is a vertex-bicolored ribbon graph. By definition of the counting function $\mathcal{P}_G$, for $(L, L') \in \mathbb{Z}^k \times \mathbb{Z}^l$, $\mathcal{P}_G(L, L')$ is equal to the number of *integer* solutions $w = \{w_e\}_{e \in E(G)}$ to the following system:

$$\begin{cases} w \in \mathrm{vp}_G^{-1}(L, L') \\ w_e > 0, \ \ e \in E(G). \end{cases} \tag{2.5}$$

Lemma 2.7 states that for $e \in E(G) \setminus B(G)$ the linear function w_e is not constant on $\mathrm{vp}_G^{-1}(L, L')$. Hence one can define the following polytope (i.e. a bounded polyhedron) in $\mathrm{vp}_G^{-1}(L, L')$:

$$M_G(L, L') = \{w \in \mathrm{vp}_G^{-1}(L, L') : w_e \geq 0, e \in E(G) \setminus B(G)\}, \tag{2.6}$$

which we call the *polytope of metrics* of G *with vertex perimeters* (L, L').

On the contrary, the linear functions $w_e, e \in B(G)$ are constant on $\mathrm{vp}_G^{-1}(L, L')$ with value $f_e(L, L')$. Then it follows from (2.5) that for $(L, L') \in H_{k,l} \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$:

$$\mathcal{P}_G(L, L') = \left(\prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0} \right) \cdot |\operatorname{int} M_G(L, L') \cap \mathbb{Z}^{E(G)}|, \tag{2.7}$$

where $\mathbf{1}$ denotes the indicator function and int denotes the interior relative to $\mathrm{vp}_G^{-1}(L, L')$.

In other words, $\mathcal{P}_G(L, L')$ is equal to the number of integer points in the interior of the polytope of metrics $M_G(L, L')$, for (L, L') in a certain (polyhedral) subset of $H_{k,l}$ given by the indicator functions. For other values of (L, L') it is identically zero.

Suppose now that (L, L') varies inside a fixed open cell C of the polyhedral subdivision $\mathcal{PS}_{k,l}$. Then, for each wall $W \in \mathcal{W}_{k,l}$, the point (L, L') is either always belongs to W or never belongs to it. In particular, the values of the indicator functions in (2.7) remain constant (since the kernels of $f_e(L, L')$ are certain walls).

Thus, it is enough to prove that $|\operatorname{int} M_G(L, L') \cap \mathbb{Z}^{E(G)}|$ depends polynomially on (L, L'). The latter statement will follow from the following general Theorem 2.18 coming from the theory of enumeration of integer points in polyhedra.

For a polyhedron P and a point p in $\mathbb{R}^d$ the *cone of feasible directions to P at p* is defined as

$$\operatorname{fcone}(P, p) = \{v \in \mathbb{R}^d : p + \varepsilon v \in P \text{ for some } \varepsilon > 0\}.$$

For example, if $P \subset \mathbb{R}^2$ is a two-dimensional convex polygon and $p \in \mathbb{R}^2$, the $\operatorname{fcone}(P, p)$ is: $\mathbb{R}^2$ if p is an interior point of P; half-space if p lies in the interior of a side of P; an acute cone if p is a vertex of P; empty set if p does not belong to P (see Figure 2.2, right).

Theorem 2.18 (Theorem 18.4 in [Bar08])**.** *Let $\{P_\alpha : \alpha \in A\}$ be a family of d-dimensional polytopes in $\mathbb{R}^d$ with vertices $v_1(\alpha), \dots, v_n(\alpha)$ such that $v_i(\alpha) \in \mathbb{Q}^d$ and the cones of feasible directions at $v_i(\alpha)$ do not depend on α:*

$$\operatorname{fcone}(P_\alpha, v_i(\alpha)) = \mathrm{const}_i, \;\; i = 1, \dots, n.$$

Suppose also that there are vectors $u_1, \dots, u_n \in \mathbb{Q}^d$ such that

$$v_i(\alpha) - u_i \in \mathbb{Z}^d, \;\; \alpha \in A, \;\; i = 1, \dots, n.$$

Then there exists a polynomial $p : (\mathbb{R}^d)^n \to \mathbb{R}$ such that

$$|\operatorname{int} P_\alpha \cap \mathbb{Z}^d| = p(v_1(\alpha), \dots, v_n(\alpha)).$$

In order to apply Theorem 2.18 in our situation, we have to identify the vertices of the polytope of metrics $M_G(L; L')$ and the cones of feasible directions at these vertices, and show that the latter do not change when $(L; L')$ changes inside the cell C. This is done in section 2.3.1.

Non-bipartite graphs

Analogously, for a non-bipartite ribbon graph $G \in \mathcal{RG}^*_{g,n}$ we define the following *polytope of metrics* in $\mathrm{vp}_G^{-1}(L)$:

$$M_G(L) = \{w \in \mathrm{vp}_G^{-1}(L) : w_e \geq 0, e \in E(G) \setminus S(G)\}. \tag{2.8}$$

We then have, for $L \in \mathbb{Z}^n$,

$$\mathcal{N}_G(L) = \left(\prod_{e \in S(G)} \mathbf{1}_{f_e(L)>0} \right) \cdot |\operatorname{int} M_G(L) \cap \mathbb{Z}^{E(G)}|. \tag{2.9}$$

Again, the values of the indicator functions in (2.9) remain constant when L varies inside a fixed open cell C of $\mathcal{PS}_n$. The polynomiality of the term $|\operatorname{int} M_G(L) \cap \mathbb{Z}^{E(G)}|$ is then proved by applying Theorem 2.18. The condition that L belongs to a fixed coset of $2\mathbb{Z}^n$ in $\mathbb{Z}^n$ is used to show that the vertices of $M_G(L)$ have fixed (rational) fractional parts. The complete proof is given in section 2.3.2.

2.2.4 Top-degree terms, positive ribbon graphs

For any $G \in \mathcal{RG}^*_{g,(k,l)}$ and any open cell C in $\mathcal{PS}_{k,l}$ denote by $top_C(\mathcal{P}_G)$ the top degree term of the restriction of $\mathcal{P}_G$ to C. Define analogously $top_C(\mathcal{N}_G)$ for any $G \in \mathcal{RG}^*_{g,n}$ and any open cell C in $\mathcal{PS}_n$.

Remark 2.19. *$top_C(\mathcal{P}_G)$ ($top_C(\mathcal{N}_G)$ respectively) is a polynomial globally defined on the affine hull of C. Its values outside of C may have nothing to do with the counting functions $\mathcal{P}_G$ ($\mathcal{N}_G$ respectively).*

We start with several observations about Theorem 2.18.

Remark 2.20. *In Theorem 2.18 the polynomial p is of degree d. Its top-degree term p_{top} gives the volume of P_α with respect to the Lebesgue measure on $\mathbb{R}^d$ normalized so that the covolume of the lattice $\mathbb{Z}^d$ is equal to 1.*

Indeed, there is an integer n such that the dilated polytope nP_α has integer vertices. The family of polytopes $P_\alpha, (1+n)P_\alpha, (1+2n)P_\alpha, \ldots$ satisfies the conditions of Theorem 2.18 and can be adjoined to the family $\{P_\alpha\}$. So

$$|\operatorname{int}((1+kn) \cdot P_\alpha) \cap \mathbb{Z}^d| \sim (kn)^d \cdot \operatorname{Vol}(P_\alpha), \;\; k \to \infty$$

is equal to

$$p((1+kn) \cdot v_1(\alpha), \ldots, (1+kn) \cdot v_n(\alpha)) \sim (kn)^{\deg(p)} \cdot p_{top}(v_1(\alpha), \ldots, v_n(\alpha)), \; k \to \infty,$$

and so $\deg(p) = d$ and $p_{top}(v_1(\alpha), \ldots, v_n(\alpha)) = \operatorname{Vol}(P_\alpha)$.

The following is a much less obvious observation, see [Bar08, Chapter 9].

Remark 2.21. *In Theorem 2.18, the top-degree term p_{top} of p gives the volume of any polytope with the same cones of feasible directions at the vertices (not just those whose vertices are rational).*

In the course of the proof of Proposition 2.16 we actually prove that *all* polytopes $M_G(L, L')$ for (L, L') in a fixed open cell of $\mathcal{PS}_{k,l}$ have the same cones of feasible directions at their vertices (Lemma 2.25). Analogous statement is true in the non-bipartite case (Lemma 2.27).

Equip each $\mathrm{vp}_G^{-1}(L, L')$ with its Lebesgue volume form normalized in such a way that the covolume of the integer lattice in any tangent space $\mathrm{vp}_G^{-1}(0, 0)$ is equal to 1.

Then we can apply Remarks 2.20 and 2.21 to the expressions (2.7) and (2.9) for the functions $\mathcal{P}_G$ and $\mathcal{N}_G$ to obtain

Lemma 2.22. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ and let C be an open cell in $\mathcal{PS}_{k,l}$. Then for all $(L, L') \in C$*

$$top_C\left(\mathcal{P}_G\right)(L, L') = \left(\prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0}\right) \cdot \mathrm{Vol}\, M_G(L, L').$$

Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph and let C be an open cell in $\mathcal{PS}_n$. Then for all $L \in C$

$$top_C\left(\mathcal{N}_G\right)(L) = \left(\prod_{e \in S(G)} \mathbf{1}_{f_e(L)>0}\right) \cdot \mathrm{Vol}\, M_G(L).$$

□

Finally, we introduce the following useful terminology.

Definition 2.23 (Positive ribbon graphs)**.** A ribbon graph $G \in \mathcal{RG}^*_{g,(k,l)}$ is *positive at a point* $(L, L') \in H_{k,l}$ if $f_e(L, L') > 0$ for all $e \in B(G)$ and $\mathrm{Vol}\, M_G(L, L') > 0$.

Likewise, a non-bipartite ribbon graph $G \in \mathcal{RG}^*_{g,n}$ is *positive at a point* $L \in \mathbb{R}^n$ if $f_e(L) > 0$ for all $e \in S(G)$ and $\mathrm{Vol}\, M_G(L) > 0$.

From the preceding discussion it follows that for any open cell C of the subdivision $\mathcal{PS}_{k,l}$ or $\mathcal{PS}_n$, a ribbon graph G is either positive at all points of C or at no point of C. We will thus also say that G is *positive on* C if it is positive at some (any) point of C.

Note also that, by Lemma 2.22, G is positive on C if and only if $top_C\left(\mathcal{P}_G\right)$ (or $top_C\left(\mathcal{N}_G\right)$) is not identically zero.

2.2.5 Degenerations of ribbon graphs and the jump of the top-degree terms on the walls

In this section we describe a general strategy that allows, in certain cases, to compute the top-degree term of a counting function on an open cell of

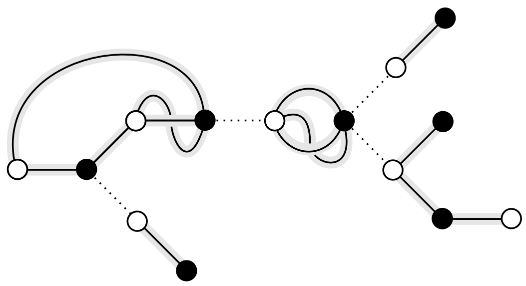


Figure 2.3: A vertex-bicolored ribbon graph which has degenerated into a disconnected graph with 4 components. The bridges of zero length are dashed. They connect the components together into a tree-like structure.

positive codimension, provided we know its top-degree term on an adjacent open cell of codimension zero. This strategy will be implemented for several different families of ribbon graphs in later chapters: vertex-bicolored plane trees (Proposition 3.11), general vertex-bicolored ribbon graphs with one boundary (Proposition 4.13), non-bipartite ribbon graphs with odd degrees of faces (section 7.1.1). We explain here only the bipartite case, the non-bipartite case is similar.

Suppose given, for all $g \geq 0$, $k, l \geq 1$, a subfamily of ribbon graphs $\mathcal{G}_{g,(k,l)} \subset \mathcal{RG}^*_{g,(k,l)}$. Let

$$\mathcal{F}_{g,(k,l)}(L, L') = \sum_{G \in \mathcal{G}_{g,(k,l)}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L, L')$$

be the corresponding counting function. It follows from Proposition 2.16 that $\mathcal{F}_{g,(k,l)}$ is polynomial in L, L' when restricted to any open cell of $\mathcal{PS}_{k,l}$.

Assume that we know the expression for the top-degree term of $\mathcal{F}_{g,(k,l)}$ on some highest-dimensional open cell C of $\mathcal{PS}_{k,l}$. How can one compute the top-degree term of $\mathcal{F}_{g,(k,l)}$ on some adjacent, lower-dimensional open cell $\widetilde{C}$?

First note that the polytopes $M_G(L, L')$ depend continuously on (L, L'). So the volumes $\operatorname{Vol} M_G(L, L')$ are continuous as well. Hence, the first part of Lemma 2.22 allows to conclude that for any $(\widetilde{L}, \widetilde{L}') \in \widetilde{C}$ the jump

$$top_C \left(\mathcal{F}_{g,(k,l)}\right)(\widetilde{L}, \widetilde{L}') - top_{\widetilde{C}} \left(\mathcal{F}_{g,(k,l)}\right)(\widetilde{L}, \widetilde{L}') \tag{2.10}$$

is equal to the sum of contributions of graphs G such that G is positive on C and G possesses at least one bridge whose length becomes zero when $(L, L') \to (\widetilde{L}, \widetilde{L}')$.

Such ribbon graphs G whose contribution to (2.10) is non-zero have the following structure (see Figure 2.3). They consist of several ribbon graphs

$G_1, \ldots, G_m$ connected together into a tree-like structure by the bridges of G whose lengths are zero at $(\widetilde{L}, \widetilde{L}')$. Note that the polytope $M_G(\widetilde{L}, \widetilde{L}')$ is just the product $\prod_{i=1}^m M_{G_i}(\widetilde{L}|_i, \widetilde{L}'|_i)$, where $(\widetilde{L}|_i, \widetilde{L}'|_i)$ are the vertex perimeters of vertices of the graph G_i . For the graph G to contribute positively to (2.10) we need $\operatorname{Vol} M_G(\widetilde{L}, \widetilde{L}') = \prod_{i=1}^m \operatorname{Vol} M_{G_i}(\widetilde{L}|_i, \widetilde{L}'|_i) > 0$. It means that each graph G_i should be positive at $(\widetilde{L}|_i, \widetilde{L}'|_i)$. The distribution of vertex labels between the graphs G_i is governed by the linear forms $f_e(L, L')$ for the zero-length bridges e. Such possible linear forms are in turn determined by the cell $\widetilde{C}$.

We say that

G degenerates into $G_1 \sqcup \ldots \sqcup G_m$ when $(L, L') \to (\widetilde{L}, \widetilde{L}')$.

If the graphs G_i belong to some families $\mathcal{G}_{g_i,(k_i,l_i)}$, this may give a recursive procedure for the computation of $top_{\widetilde{C}}\left(\mathcal{F}_{g,(k,l)}\right)$. Indeed, since $\operatorname{Vol} M_G(\widetilde{L}, \widetilde{L}') = \prod_{i=1}^m \operatorname{Vol} M_{G_i}(\widetilde{L}|_i, \widetilde{L}'|_i)$, one can try to express the jump (2.10) in terms of the products $\prod_{i=1}^m top_{C_i}\left(\mathcal{F}_{g_i,(k_i,l_i)}\right)$. One is then naturally led to the following question:

how many graphs G which are positive on C degenerate into $G_1 \sqcup \ldots \sqcup G_m$ when passing to $\widetilde{C}$?

This is a non-trivial question, since not all ways of connecting the G_i into a tree-like structure by zero-length edges produce a graph G that is positive on C. One has to characterize and count such ways. We will solve such problem in the case of vertex-bicolored graphs with one boundary (proof of Proposition 3.11) and for the graphs with odd face degrees (Proposition 7.6).

2.3 Proofs

2.3.1 Proof of Proposition 2.16

Lemma 2.24. *Let $G \in \mathcal{RG}^*_{g,(k,l)}$ be a ribbon graph and let $(L, L') \in H_{k,l}$. Then $w \in \mathrm{vp}_G^{-1}(L, L')$ is a vertex of $M_G(L, L')$ if and only if $w_e \geq 0$ for all $e \in E(G) \setminus B(G)$ and the edges in $F = \{e \in E(G) \setminus B(G) : w_e > 0\}$ form a forest (i.e. a disjoint union of trees).*

The cone of feasible directions to $M_G(L, L')$ at such vertex w is given by the system

$$\begin{cases} v \in \mathrm{vp}_G^{-1}(0,0), \\ v_e \geq 0, \ \ e \in E(G) \setminus (B(G) \cup F). \end{cases}$$

In particular, it only depends on F and not on L, L'.

Proof. Suppose w is such that $w_e \geq 0$ for all $e \in E(G) \setminus B(G)$ and the edges in $F = \{e \in E(G) \setminus B(G) : w_e > 0\}$ form a forest. The first condition ensures that $w \in M_G(L, L')$, by definition of $M_G(L, L')$. The second condition ensures that w is not a midpoint of a segment (not reduced to a point) whose endpoints lie in $M_G(L, L')$. Indeed, suppose $w = (w' + w'')/2$ with $w', w'' \in M_G(L, L')$. Since $w_e = 0, w'_e \geq 0, w''_e \geq 0$ for $e \in E(G) \setminus (B(G) \cup F)$, necessarily $w'_e = w''_e = 0$ for $e \in E(G) \setminus (B(G) \cup F)$. But then, by Lemma 2.5, the weights $w'_e, w''_e, e \in B(G) \cup F$ of w' and w'' are uniquely determined ($B(G) \cup F$ forms a collection of trees) and are equal to the corresponding weights of w. Hence $w' = w'' = w$ and w is an extreme point of $M_G(L, L')$, hence a vertex.

Conversely, let w be a vertex of $M_G(L, L')$. Then $w_e \geq 0$ for all $e \in E(G) \setminus B(G)$ because $w \in M_G(L, L')$. Vertices of $M_G(L, L')$ are exactly its extreme points, but if there were a cycle of edges of positive weight in w, one would be able to modify the weights in this cycle by alternately adding and subtracting ε or $-\varepsilon$ to/from the weights of consecutive edges of this cycle, for some small $\varepsilon > 0$, and write w as a midpoint of these two modifications, which still belong to $M_G(L, L')$. So F indeed forms a forest.

The second claim follows from the definition of the cone of feasible directions and the defining system for $M_G(L, L')$ (the weights of edges in $E(G) \setminus (B(G) \cup F)$ can only be perturbed in the positive direction, while the weights of edges in F can be perturbed arbitrarily). □

For a vertex w of $M_G(L, L')$ we call the set $F \subset E(G) \setminus B(G)$ as in Lemma 2.24 the *support* of w.

Lemma 2.25. *Fix a ribbon graph $G \in \mathcal{RG}^*_{g,(k,l)}$ and an open cell C in $\mathcal{PS}_{k,l}$.*

There exist subsets $F_1, \dots, F_n \subset E(G) \setminus B(G)$ each forming a forest, such that each polytope $M_G(L, L')$ with $(L, L') \in C$ has n vertices $v_1(L, L'), \dots, v_n(L, L')$ with supports $F_1, \dots, F_n$ respectively. For each i the coordinates of $v_i(L, L')$ are either identically zero or are linear functions (of L, L') from $\mathcal{L}_{k,l}$. If $(L, L') \in \mathbb{Z}^k \times \mathbb{Z}^l$, then all of the vertices of $M_G(L, L')$ are integral.

Moreover, for each i the cone of feasible directions $\mathrm{fcone}(M_G(L, L'), v_i(L, L'))$ *is constant (does not depend on L, L').*

Proof. Consider a subset $F \subset E(G) \setminus B(G)$ which forms a forest. Then $B(G) \cup F$ also forms a forest, i.e. a collection of trees. So by Lemma 2.5 there exists a (unique) weight function w on G such that $w_e = 0$ for $e \notin B(G) \cup F$ if and only if for each constituent tree of $B(G) \cup F$ the sums of perimeters of its black and its white vertices are equal (condition 1). If such w exists, then by Lemma 2.24 F is a support of a vertex of $M_G(L, L')$ if and only if $w_e > 0$ for $e \in F$ (condition 2).

Note that conditions 1 and 2 are equivalent to the fact that several linear functions from $\mathcal{L}_{k,l}$ are zero (for condition 1) or positive (for condition 2, because edge weights for trees are given by linear functions from $\mathcal{L}_{k,l}$ by Lemma 2.4).

By definition of the polyhedral subdivision $\mathcal{PS}_{k,l}$, when (L, L') stays in C, the signs ($+$, $-$ or 0) of all linear forms in $\mathcal{L}_{k,l}$ remain constant (we do not leave or enter any new walls from $\mathcal{W}_{k,l}$). So for each F conditions 1 and 2 are either satisfied everywhere or nowhere on C. Hence each F is a support of a (unique) vertex of $M_G(L, L')$ either for all $(L, L') \in C$ or for none of them. This proves the first claim.

It follows from the discussion above that the edge weights at the vertices of $M_G(L, L')$ are either identically zero or are linear functions from $\mathcal{L}_{k,l}$. Integrality claim follows since linear functions from $\mathcal{L}_{k,l}$ have integer coefficients. Finally, the cone of feasible directions at a vertex is determined by its support by Lemma 2.24. □

We now pass to the proof of Proposition 2.16.

Proof of Proposition 2.16. Recall the formula (2.7) for $\mathcal{P}_G(L, L')$. As in the proof of Lemma 2.25, when (L, L') stays in C, the signs ($+$, $-$ or 0) of all linear forms in $\mathcal{L}_{k,l}$ remain constant. It means that the product of indicator functions in (2.7) is constant on C.

By Lemma 2.25, all the polytopes $M_G(L, L')$ with $(L, L') \in C \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$ have the same number of vertices, all these vertices are integral, and the cones of feasible directions at the corresponding vertices are the same. In particular, all the polytopes have the same dimension. If their common dimension is less then the dimension of $\mathrm{vp}_G^{-1}(L, L')$, their interiors are empty and the second term of (2.7) is identically zero. Otherwise, note that $\mathrm{vp}_G^{-1}(L, L') \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$ is a full rank lattice in $\mathrm{vp}_G^{-1}(L, L')$ by Lemma 2.8. This allows us to apply Theorem 2.18 to the second term, which show that it is a polynomial of degree $\dim \mathrm{vp}_G^{-1}(L, L') = |V(G)| - |E(G)| + 1$ in the coordinates of the vertices, which are themselves linear functions of L, L' (by Lemma 2.25), which proves Proposition 2.16.

Note that, formally, one cannot apply Theorem 2.18 to a family of polytopes $M_G(L, L')$ belonging to different parallel affine subspaces $\mathrm{vp}_G^{-1}(L, L')$. However, one can first identify each $\mathrm{vp}_G^{-1}(L, L')$ with $\mathrm{vp}_G^{-1}(0, 0)$ by a translation. The translation vector (depending linearly on L, L') can be chosen as the unique weight function w on G such that $w_e = 0$ for $e \notin E(T)$ for some fixed spanning tree T of G. For $(L, L') \in \mathbb{Z}^k \times \mathbb{Z}^l$ this vector will be integral and so the integral lattice of $\mathrm{vp}_G^{-1}(L, L')$ is identified by this translation with the integral lattice of $\mathrm{vp}_G^{-1}(0, 0)$ and we can apply Theorem 2.18. □

2.3.2 Proof of Proposition 2.17

The proof of Proposition 2.17 is analogous to the proof of Proposition 2.16, so we skip the details, concentrating on the parts that are different.

Lemma 2.26. *Let $G \in \mathcal{RG}^*_{g,n}$ be a non-bipartite ribbon graph and let $L \in \mathbb{R}^n$. Then $w \in \mathrm{vp}_G^{-1}(L)$ is a vertex of $M_G(L)$ if and only if $w_e \geq 0$ for all $e \in E(G) \setminus S(G)$ and the edges in $F = \{e \in E(G) \setminus S(G) : w_e > 0\}$ form a disjoint union of trees and/or OOC graphs.*

The cone of feasible directions to $M_G(L)$ at such vertex w is given by the system

$$\begin{cases} v \in \mathrm{vp}_G^{-1}(0), \\ v_e \geq 0, \;\; e \in E(G) \setminus (S(G) \cup F). \end{cases}$$

In particular, it only depends on F and not on L.

□

We skip the proof, which is analogous to the proof of Lemma 2.24.

For a vertex w of $M_G(L)$ we call the set $F \subset E(G) \setminus S(G)$ as in Lemma 2.26 the *support* of w.

Lemma 2.27. *Fix a non-bipartite ribbon graph $G \in \mathcal{RG}^*_{g,n}$ and an open cell C of $\mathcal{PS}_n$.*

There exist subsets $F_1, \ldots, F_m \subset E(G) \setminus S(G)$ each forming a disjoint union of trees and/or OOC graphs, such that each polytope $M_G(L)$ with $L \in C$ has m vertices $v_1(L), \ldots, v_m(L)$ with supports $F_1, \ldots, F_m$ respectively.

For each i the coordinates of $v_i(L)$ are either identically zero or are linear functions (of L) from $\mathcal{L}_n \cup \frac{1}{2}\mathcal{L}_n$.

For each i the cone of feasible directions $\mathrm{fcone}(M_G(L), v_i(L))$ *is constant (does not depend on L).*

Proof. By Lemma 2.26, a subset $F \subset E(G) \setminus S(G)$ forming a disjoint union of trees and/or OOC graphs is a support of some vertex of $M_G(L)$ if and only if the unique weight function w_F on F with vertex perimeters L is positive. By Lemma 2.10, the weights of edges in w_F are given by linear forms in $\mathcal{L}_n \cup \frac{1}{2}\mathcal{L}_n$. When L stays inside C, the signs ($+, -$ or 0) of all these functions remain constant. Hence an F corresponds to a vertex of $M_G(L)$ either for all $L \in C$ or for none. □

Proof of Proposition 2.17. Fix a non-bipartite ribbon graph $G \in \mathcal{RG}^*_{g,n}$, an open cell C of $\mathcal{PS}_n$ and a coset Λ of $2\mathbb{Z}^n$ in $\mathbb{Z}^n$.

If for $L \in \Lambda$ we have $L_1 + \ldots + L_n = 1 \pmod 2$, then $\mathcal{N}_G(L)$ is zero identically.

Otherwise, be Lemma 2.15, $\mathrm{vp}_G^{-1}(L) \cap \mathbb{Z}^{E(G)}$ is a full-dimensional lattice in $\mathrm{vp}_G^{-1}(L)$. Consider the family of polytopes $\{M_G(L), L \in C \cap \Lambda\}$. By Lemma 2.27, all these polytopes have the same cones of feasible directions at corresponding vertices. Moreover, the corresponding coordinates of these vertices are either integer or half-integer for all $L \in C \cap \Lambda$ (because, for L in a fixed coset of $2\mathbb{Z}^n$ in $\mathbb{Z}^n$, the value of any $f \in \mathcal{L}_n \cup \frac{1}{2}\mathcal{L}_n$ is either always integer or always half-integer). One can thus apply Theorem 2.18 to this family of polytopes. We conclude by noticing that for $L \in C$ the product of indicators in (2.9) remains constant. $\square$

Chapter 3

One-vertex graphs

In this chapter we study the functions $\mathcal{P}^g_{k,l}(L;L')$, which count face-bicolored metric ribbon graphs with one vertex and given perimeters of the boundary components. We start by proving that the top-degree terms of their restrictions to W° are polynomial for any subspace $W \in \overline{\mathcal{W}_{k,l}}$ (Theorem 3.2). We then give explicit formulas for the top-degree terms of $\mathcal{P}^g_{k,l}$ on $H^\circ_{k,l}$ (Proposition 3.3) as well as the top-degree term of $\mathcal{P}^g_{n,n}$ on V°_n, where $V_n \in \overline{\mathcal{W}_{n,n}}$ is the subspace of $\mathbb{R}^n \times \mathbb{R}^n$ defined by the equations $L_i = L'_i$, $i = 1, \ldots, n$ (Theorem 3.4). This last explicit expression is then used to deduce the explicit generating function for the contributions of n-cylinder square-tiled surfaces to the volumes of the minimal strata $\mathcal{H}(2g-2)$ of Abelian differentials (Theorem 3.5).

The material in this chapter is drawn from my paper [Yak23].

3.1 Statements of results

Recall the following notations.

$\mathcal{E}_{g,k,l}$ denotes the set of face-bicolored ribbon graphs of genus g with k black and l white labeled faces and with one vertex. The dual family $\mathcal{E}^*_{g,k,l}$ consists of vertex-bicolored ribbon graphs of genus g with k black and l white labeled vertices and with one boundary component. The counting function of the family $\mathcal{E}_{g,k,l}$ is denoted by $\mathcal{P}^g_{k,l}(L;L')$. Recall also the duality Convention 1.7.

Recall finally the definitions of and the notations for the subspace $H_{k,l} \subset \mathbb{R}^k \times \mathbb{R}^l$, the polyhedral subdivision $\mathcal{PS}_{k,l}$ of $H_{k,l}$, the set of walls $\mathcal{W}_{k,l}$ and the set of their intersections $\overline{\mathcal{W}_{k,l}}$, introduced in section 2.2.1.

First of all, the general nature of the counting functions is given by

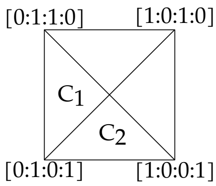


Figure 3.1: Projectivization of the cone $H^+_{2,2}$

Proposition 3.1. *For every $W \in \overline{\mathcal{W}_{k,l}}$ and every connected component C of $H^+_{k,l} \cap W^\circ$, the function $\mathcal{P}^g_{k,l}(L, L')$ is either identically zero or is given by a polynomial in L, L' of degree $2g$ for $(L, L') \in C \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$.*

Proof. Any connected component C of $H^+_{k,l} \cap W^\circ$ is an open cell of the polyhedral subdivision $\mathcal{PS}_{k,l}$ of $H_{k,l}$. Thus, by Proposition 2.16, the contribution of any $G \in \mathcal{E}^*_{g,k,l}$ to $\mathcal{P}^g_{k,l}$ on C is either zero or a polynomial of degree

$$|E(G)| - |V(G)| + 1 = |F(G)| + (2g - 2) + 1 = 2g.$$

□

Let us consider an explicit example, to illustrate that the polynomials on different connected components are in general different. Let $g = 1$, $k = l = 2$. Then the cone $H^+_{2,2} = \{L_1 + L_2 = L'_1 + L'_2\} \cap \mathbb{R}^4_{>0}$ is 3-dimensional. There are two walls in $\mathcal{W}_{2,2}$ that intersect the cone transversely: $L_1 = L'_1$ and $L_1 = L'_2$. These subdivide the cone into 4 open cells. Projectivizing, the cone becomes a quadrangle, while the walls become its diagonals (see Figure 3.1).

Proposition 3.1 applied to $W = H_{2,2}$ states that $\mathcal{P}^1_{2,2}$ is a quadratic polynomial on each of the four cells. One can compute these polynomials by hand (this is a bit tedious since there are 18 ribbon graphs contributing on each cell). We give here the results for the cells $C_1 = \{L_1 < L'_1, L_1 < L'_2\}$ and $C_2 = \{L_1 > L'_1, L_1 < L'_2\}$.

$$\mathcal{P}^1_{2,2}|_{C_1}(L, L') = (L_1^2 + L_2^2 + {L'_1}^2 + {L'_2}^2) - 6L_1 - 10L_2 + 18,$$
$$\mathcal{P}^1_{2,2}|_{C_2}(L, L') = (L_1^2 + L_2^2 + {L'_1}^2 + {L'_2}^2) - 6L'_1 - 10L'_2 + 18.$$

The polynomials are different since their difference $6L'_1+10L'_2-6L_1-10L_2$ is not divisible by $L_1 + L_2 - L'_1 - L'_2$ (these polynomials are only defined on $H_{2,2}$). However, the top-degree terms do coincide. It turns out that this is a general phenomenon.

Theorem 3.2. *For all $g \geq 0, k, l \geq 1$ and every $W \in \overline{\mathcal{W}_{k,l}}$ there exists a homogeneous polynomial P^g_W in the variables L, L' of degree $2g$ (or identically zero) such that for all $(L, L') \in \mathbb{Z}^k \times \mathbb{Z}^l$ belonging to $H^+_{k,l} \cap W^\circ$ we have*

$$\mathcal{P}^g_{k,l}(L, L') = P^g_W(L, L') + \textit{terms of degree at most } 2g - 1.$$

By Proposition 3.1, the "terms of degree at most $2g-1$" are given by a polynomial of degree at most $2g-1$ on each connected component of $H^+_{k,l} \cap W^\circ$.

In other terms, for every connected component C of $H^+_{k,l} \cap W^\circ$ and for all $(L, L') \in W$ we have, independently of C,

$$top_C(\mathcal{P}^g_{k,l})(L, L') = P^g_W(L, L').$$

The proof of this theorem occupies section 3.2. Note that when $g = 0$, each P^0_W is just a constant polynomial. We first give a simple bijective proof for $g = 0$ (section 3.2.1), which involves flips of edges in bipartite plane trees. We then deduce the case $g > 0$ from the case $g = 0$ with the help of a result from the theory of combinatorial maps, the Chapuy-Féray-Fusy bijection (Theorem 3.8).

Note that for fixed $g > 0$ and $W \in \overline{\mathcal{W}_{k,l}}$, the expression for $P^g_W(L, L')$ as a polynomial of (L, L') is not unique, because we are only considering its values on the subspace W. It is only uniquely defined modulo the linear functions defining W.

Nevertheless, the proof of Theorem 3.2 also gives an explicit expression for P^g_W with $g > 0$ as a polynomial in L, L', with the property that the coefficients are the (normalized) constants P^0_W for various subspaces W (see Theorem 3.6 for a precise statement). In particular, we deduce in section 3.3 the following explicit expressions for P^g_W when $W = H_{k,l} \in \overline{\mathcal{W}_{k,l}}$ and $W = V_n := \{L_1 = L'_1, \dots, L_n = L'_n\} \in \overline{\mathcal{W}_{n,n}}$.

Proposition 3.3. *For all $g \geq 0$, $k, l \geq 1$ we have*

$$P^g_{H_{k,l}}(L, L') = \frac{(k+l+2g-2)!}{2^{2g}} \cdot \sum_{\substack{b_1+\ldots+b_k+w_1+\ldots+w_l=g \\ b_i, w_i \geq 0}} \prod_{i=1}^{k} \frac{L_i^{2b_i}}{(2b_i+1)!} \cdot \prod_{j=1}^{l} \frac{{L'_j}^{2w_j}}{(2w_j+1)!}.$$

Theorem 3.4. *For all $g \geq 0, n \geq 1$ we have*

$$P^g_{V_n}(L, L) = 2^n \cdot \sum_{\substack{s_1+\ldots+s_n=g+n \\ s_i \geq 1}} p_{2s_1,\ldots,2s_n} \frac{L_1^{2s_1-2}}{(2s_1)!} \cdots \frac{L_n^{2s_n-2}}{(2s_n)!},$$

where the numbers $p_{2s_1,\ldots,2s_n} \in \mathbb{Z}_{>0}$ are part of a bigger collection of numbers $p_{s_1,\ldots,s_n} \in \mathbb{Z}_{>0}$ ($n \geq 1, s_i \geq 2$) that are symmetric in the indices s_i, and whose generating function in an infinite number of variables

$$\mathcal{T}(t, t_2, t_3, \ldots) = 1 + \sum_{s,n \geq 1} (s-1) t^s \frac{1}{n!} \sum_{\substack{s_1+\ldots+s_n=s \\ s_i \geq 2}} p_{s_1,\ldots,s_n} t_{s_1} \cdots t_{s_n}$$

satisfies the following relation for all $k \geq 0$:

$$\frac{1}{k!}[t^k]\mathcal{T}(t, t_2, t_3, \ldots)^k = [t^k] \exp\left(\sum_{i\geq 2} t_i t^i\right). \tag{3.1}$$

Using Lagrange inversion, (3.1) can be rewritten equivalently as

$$\mathcal{T}(t, t_2, \ldots) = \frac{t}{Q^{-1}(t, t_2, \ldots)}, \; Q(t, t_2, \ldots) = t \cdot \exp\left(\sum_{k=1}^{\infty}(k-1)! b_k(t_2, \ldots) t^k\right),$$

where $b_k(t_2, \ldots) = [t^k] \exp\left(\sum_{i\geq 2} t_i t^i\right)$ and functional inversion is with respect to the variable t. In particular, the numbers $p_{s_1,\ldots,s_n}$ can be effectively computed. We present in Table 3.1 the values of some $p_{2s_1,\ldots,2s_n}$ with small indices.

p_2	p_4	$p_{2,2}$	p_6	$p_{4,2}$	$p_{2,2,2}$	p_8	$p_{6,2}$	$p_{4,4}$	$p_{4,2,2}$	$p_{2,2,2,2}$
1	2	1	24	18	11	720	600	684	486	335

Table 3.1: Values of $p_{2s_1,\ldots,2s_n}$ with $s_1 + \ldots + s_n \leq 4$.

Recall from section 1.3 the definitions of square-tiled surfaces, their cylinder decomposition and the strata of Abelian differentials. Recall also the formula (1.2) expressing the volume of the stratum $\mathcal{H}(k)$ in terms of the asymptotic count of square-tiled surfaces in $\mathcal{H}(k)$ with at most N squares.

By analogy, one can define the contributions of n-cylinder square-tiled surfaces to this volume as

$$\mathrm{Vol}_n(k) = 2d \cdot \lim_{N\to+\infty} \frac{|\mathcal{ST}_n(\mathcal{H}(k), N)|}{N^d}. \tag{3.2}$$

where $\mathcal{ST}_n(\mathcal{H}(k), N)$ is the set of square-tiled surfaces in $\mathcal{H}(k)$ with n cylinders and at most N squares, and $d = 2g - s + 1$, with s the length of k. The existence of this limit is not obvious, see Section 1.1 in [DGZ+20] and the references therein.

Using Theorem 3.4 and the approach described in section 1.3.4, we will compute the following explicit generating series for the contributions of n-cylinder square-tiled surfaces to the volumes of the minimal strata $\mathcal{H}(2g-2)$ of Abelian differentials. This computation is performed in section 3.4.

Theorem 3.5. *The contribution* $\mathrm{Vol}_n(2g-2)$ *of n-cylinder square-tiled surfaces to the volume of the minimal stratum $\mathcal{H}(2g-2)$ is equal to* $\frac{2(2\pi)^{2g}}{(2g-1)!}a_{g,n}$, *where the numbers $a_{g,n} \in \mathbb{Q}$, and whose bivariate generating function*

$$\mathcal{C}(t, u) = 1 + \sum_{g\geq 1}\left(\sum_{n=1}^{g} a_{g,n} u^n\right)(2g-1)t^{2g}$$

satisfies for all $g \geq 0$

$$\frac{1}{(2g)!}[t^{2g}]\mathcal{C}(t,u)^{2g} = [t^{2g}]\left(\frac{t/2}{\sin(t/2)}\right)^{u}, \tag{3.3}$$

where $[t^{2g}]$ *stands for the extraction of the coefficient of the corresponding monomial.*

Using Lagrange inversion, (3.3) can be rewritten equivalently as

$$\mathcal{C}(t,u) = \frac{t}{Q^{-1}(t,u)}, \; Q(t,u) = t \cdot \exp\left(\sum_{k=1}^{\infty}(k-1)! b_k(u) t^k\right),$$

where $b_k(u) = [t^k]\left(\frac{t/2}{\sin(t/2)}\right)^u$ and functional inversion is with respect to the variable t. In particular, the numbers $a_{g,n}$ can be effectively computed. We present in Table 3.2 the values of $a_{g,n}$ for small genera g.

$g\backslash n$	1	2	3	4
1	$\frac{1}{24}$			
2	$\frac{1}{1440}$	$\frac{1}{1152}$		
3	$\frac{1}{7560}$	$\frac{1}{3840}$	$\frac{11}{82944}$	
4	$\frac{1}{13440}$	$\frac{5197}{29030400}$	$\frac{3}{20480}$	$\frac{335}{7962624}$

Table 3.2: Values of the normalized volume contributions $a_{g,n}$ for $g \leq 4$.

The particular case $u = 1$ of Theorem 3.5, which gives the generating function for the (normalized) total volumes of $\mathcal{H}(2g-2)$, was obtained by Sauvaget [Sau18, Theorem 1.6] via intersection theory. There the numbers $a_g = \sum_{n=1}^{g} a_{g,n}$ are shown to be equal to certain intersection numbers. The intersection-theoretic interpretation of the refined numbers $a_{g,n}$ (if exists) is currently unknown to the author.

Relation to Hurwitz numbers / dessins d'enfants

Face-bicolored (integer metric) ribbon graphs (or their duals) are equivalent to covers of the sphere ramified over three points, and are also known as *dessins d'enfants*, see [LZ04, Chapter 2]. The numbers of such covers with prescribed ramification profiles over the branch points are examples of *Hurwitz numbers*.

The formula from Proposition 3.3 can also be deduced (using some additional arguments, we omit the details) from Theorems 1 and 2 in [OP06], where the authors consider the corresponding Hurwitz numbers with completed cycles. Their methods however are algebraic.

One should also mention the paper [KZ15], where the authors prove the following statement about enumeration of dessins d'enfants (which we translate here into the language of ribbon graphs): the generating series for the numbers of face-bicolored (non-metric) ribbon graphs with given number of black and white faces and with given degrees of vertices, satisfies: the Virasoro constraints, an evolution equation, the KP hierarchy and the topological recursion. Note however that here the degrees of faces are not specified.

Passing to the dual

Recall the duality Convention 1.7. In the rest of this chapter we will work with the dual families $\mathcal{E}^*_{g,k,l}$. In particular, we will use the alternative definition of the counting function $\mathcal{P}^g_{k,l}$ in terms of dual graphs:

$$\mathcal{P}^g_{k,l}(L, L') = \sum_{G \in \mathcal{E}^*_{g,k,l}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L, L'),$$

where $\mathcal{P}_G(L, L')$ is the number of integer metrics on G with *vertex* perimeters given by L, L'.

3.2 Proof of Theorem 3.2

3.2.1 Case $g = 0$: counting positive trees

Note that the elements of $\mathcal{E}^*_{0,k,l}$ are vertex-bicolored plane trees with k black and l white labeled vertices. Note that every tree $G \in \mathcal{E}^*_{0,k,l}$ has no non-trivial automorphisms, i.e. $|\operatorname{Aut}(G)| = 1$. Indeed, an automorphism of G must simultaneously preserve a leaf of G (automorphisms preserve the labelling) and the order of edges along the unique boundary, so it is necessarily the identity.

By Lemma 2.5, for every tree $G \in \mathcal{E}^*_{0,k,l}$ and every $(L, L') \in H_{k,l}$ there exists a unique weight function w on G with $\operatorname{vp}_G(w) = (L, L')$. This weight function w is an integral metric if and only if $(L, L') \in H_{k,l} \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$ and all the edge weights are positive. By Lemma 2.4, the weight w_e of every edge $e \in E(G)$ is given by some linear function $f_e(L, L')$ from the set $\mathcal{L}_{k,l}$. Hence, for $(L, L') \in H_{k,l} \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$ we have

$$\mathcal{P}_G(L, L') = \prod_{e \in E(G)} \mathbf{1}_{f_e(L,L')>0}. \tag{3.4}$$

This is a special case of the formula (2.7), which we derive here again for clarity. Recall the definition of ribbon graphs positive at a point (Definition 2.23). Clearly, for any $(L, L') \in H_{k,l}$ the right-hand side of (3.4) is equal to 1 if and only if the tree G is positive at (L, L').

Combining this with $|\operatorname{Aut}(G)| = 1$, we see that the top-degree term of $\mathcal{P}^g_{k,l}$ on any open cell C of $\mathcal{PS}_{k,l}$ is the constant equal to the number of trees positive at (L, L') for any $(L, L') \in C$.

Thus, to prove Theorem 3.2 for $g = 0$ and the subspace W, we simply have to show that the number of positive trees does not change when we pass from one connected component of $H^+_{k,l} \cap W^\circ$ to another adjacent one.

Proof of Theorem 3.2 for $g = 0$. Fix $W \in \overline{\mathcal{W}_{k,l}}$. Take $V \in \overline{\mathcal{W}_{k,l}}$ such that $V \subset W$, V is of codimension 1 in W and V intersects the open cone $H^+_{k,l} \cap W$. It is enough to prove that the number of trees positive at (L, L') does not change when the point (L, L') traverses V inside W.

Consider a (small) oriented linear path $\gamma \subset H^+_{k,l} \cap W$ transversal to V. When (L, L') reaches V along γ, certain positive trees (forming a subset $\mathcal{D}^- \subset \mathcal{E}^*_{0,k,l}$) cease to be positive (some of their edges become zero-weight), and the number of positive trees decreases by $|\mathcal{D}^-|$. In turn, when (L, L') continues along γ past the point of intersection with V, some non-positive trees (forming a subset $\mathcal{D}^+ \subset \mathcal{E}^*_{0,k,l}$) become positive, and the number of positive trees increases by $|\mathcal{D}^+|$. It is now enough to show that $|\mathcal{D}^-| = |\mathcal{D}^+|$. We will do this by establishing a bijection between $\mathcal{D}^-$ and $\mathcal{D}^+$, which we now describe.

Take a tree $T^- \in \mathcal{D}^-$. When $(L, L') = \gamma \cap V$, the tree T^- has some zero-weight edges. We call these edges *bad* and the other ones *good*. Let T^0 be the disjoint union of trees formed by good edges. Every bad edge e in T^- connects two corners $c_1(e), c_2(e)$ of two trees $t_1(e), t_2(e)$ in T^0. Define *flipping* as the following procedure transforming T^- into another tree T^+: for every bad edge e, replace e by an edge joining two corners following $c_1(e)$ and $c_2(e)$ counterclockwise around $t_1(e)$ and $t_2(e)$, respectively (Figure 3.2a). If there are several bad edges emanating from the same vertex of T^- and which are consecutive for the circular order around this vertex, flipping should preserve their circular order (Figure 3.2b).

We claim that flipping is well-defined, that $T^+ \in \mathcal{D}^+$ and that flipping gives a bijection between $\mathcal{D}^-$ and $\mathcal{D}^+$.

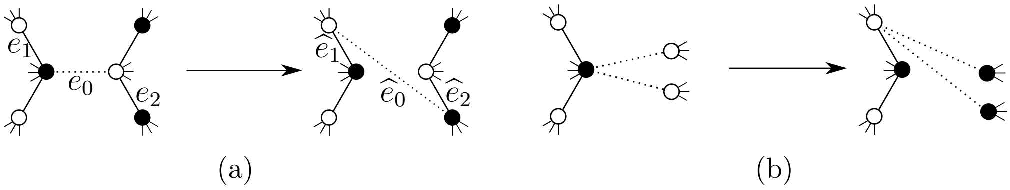


Figure 3.2: Flipping zero-weight edges.

The only obstacle to flipping is when there is a vertex of T^- which is only incident to bad edges, but this is not possible in our situation. Indeed, when $(L, L') = \gamma \cap V$, the perimeter of this vertex must be zero (being the sum of weights of bad edges). But this implies that $\gamma \cap V \notin H^+_{k,l}$, a contradiction. So flipping is well-defined.

We now show that $T^+ \in \mathcal{D}^+$. First, consider an arbitrary tree T with three of its edges e_0, e_1, e_2 arranged as in Figure 3.2a and suppose we have flipped the edge e_0 to get a tree $\widehat{T}$. For every edge e of T let $\widehat{e}$ be the corresponding edge of $\widehat{T}$. As before, for an edge e we denote by f_e the linear function from $\mathcal{L}_{k,l}$ giving its weight as a function of vertex perimeters. Lemma 2.4 implies that $f_{\widehat{e}_0} = -f_{e_0}$, $f_{\widehat{e}_1} = f_{e_1} + f_{e_0}$, $f_{\widehat{e}_2} = f_{e_2} + f_{e_0}$ and $f_{\widehat{e}} = f_e$ for all other edges e of T. In particular, if, restricted to γ, f_{e_0} changes sign from positive to negative at $\gamma \cap V$ and f_{e_1} and f_{e_2} remain positive, then $f_{\widehat{e}_0}$ changes sign from negative to positive at $\gamma \cap V$ and $f_{\widehat{e}_1}$ and $f_{\widehat{e}_2}$ remain positive. If we now apply this observation to each flip we make to get from T^- to T^+, we see that $T^+ \in \mathcal{D}^+$.

Finally, this procedure gives a bijection between $\mathcal{D}^-$ and $\mathcal{D}^+$ because one can construct an inverse map from $\mathcal{D}^+$ to $\mathcal{D}^-$ by flipping in the opposite direction the bad edges of trees from $\mathcal{D}^+$. □

As an illustration to the proof of Theorem 3.2 for $g = 0$, we present in Figure 3.3 the case $k = l = 3$, $W = \{L_1 = L'_1, L_2 = L'_2, L_3 = L'_3\}$, $V = \{L_1 = L_2\}$.

3.2.2 Case $g > 0$: a stronger result

We now complete the proof of the polynomiality of the top-degree term of the counting functions $\mathcal{P}^g_{k,l}$ (Theorem 3.2) for $g > 0$. In fact we prove a stronger statement (Theorem 3.6 below), of which Theorem 3.2 is a direct corollary. It will also allow us to get explicit expressions for the top-degree terms of $\mathcal{P}^g_{k,l}$ on some particular subspaces (see section 3.3).

Theorem 3.6. *For every $g \geq 0, k, l \geq 1$ and every $W \in \overline{\mathcal{W}_{k,l}}$ there exists a family of subspaces $U_{W,b,w} \in \overline{\mathcal{W}_{k+2g_1,l+2g_2}}$ (where $b = (b_1, \ldots, b_k)$, $w =$*

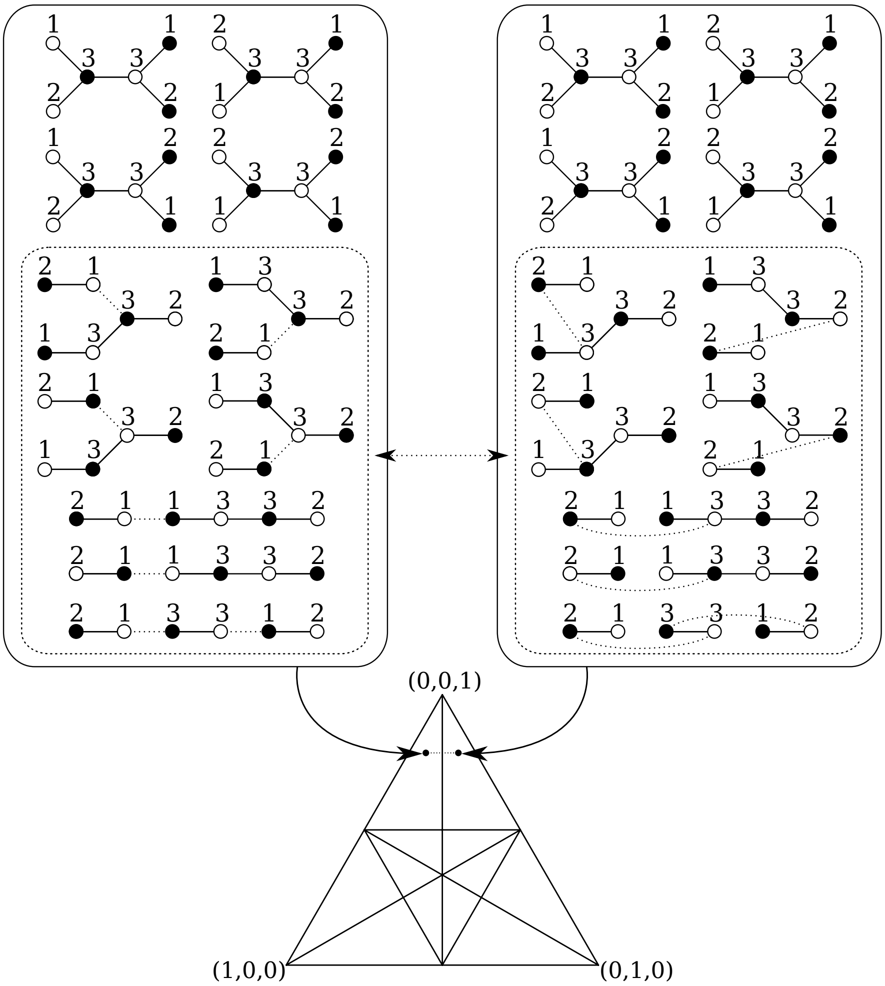


Figure 3.3: Illustration to the proof of Theorem 3.2 for $g = 0$, where $k = l = 3$, $W = \{L_1 = L'_1, L_2 = L'_2, L_3 = L'_3\}$, $V = \{L_1 = L_2\}$. At the bottom of the figure we see the projectivization of the three-dimensional cone $H^+_{k,l} \cap W = \{L_1 = L'_1, L_2 = L'_2, L_3 = L'_3, L_1 > 0, L_2 > 0, L_3 > 0\}$. The cone is divided by 6 codimension 1 subspaces from $\overline{\mathcal{W}_{k,l}}$ into 12 "cells". The number of trees positive at each point inside of each cell is the same (11 in this case). At the top of the figure are given the sets of positive trees corresponding to two particular points of the cone. Going from one of the points to the other, we must cross the subspace V. When we reach V from one side, certain trees cease to be positive — some of their edges become zero-weight (these edges are marked by dotted lines). In turn, when we continue to the other cell, certain trees become positive. The flipping procedure described in the proof provides a bijection between these sets of trees. In the figure, the corresponding trees are opposite to each other.

$(w_1,\dots,w_l)$ *are vectors of non-negative integers and* $b_1+\dots+b_k=g_1$, $w_1+\dots+w_l=g_2$*) such that for every connected component* C *of* $H^+_{k,l}\cap W^\circ$, *we have*

$$top_C\left(\mathcal{P}^g_{k,l}\right)(L,L') = 2^{-2g}\cdot \sum_{\substack{b_1+\ldots+b_k+w_1+\ldots+w_l=g\\ b_i,w_i\geq 0}} P^0_{U_{W,b,w}}\cdot\prod_{i=1}^{k}\frac{L_i^{2b_i}}{(2b_i+1)!}\cdot\prod_{j=1}^{l}\frac{{L'_j}^{2w_j}}{(2w_j+1)!}. \tag{3.5}$$

In particular, $top_C\left(\mathcal{P}^g_{k,l}\right)$ *does not depend on the connected component* C.

Recall that, by definition, $P^0_{U_{W,b,w}}$ is the top-degree term of $\mathcal{P}^0_{k+2g_1,l+2g_2}$ on $H^+_{k+2g_1,l+2g_2}\cap U^\circ_{W,b,w}$. From the previous subsection we know that this is simply a constant, equal to the number of trees in $\mathcal{E}^*_{0,k+2g_1,l+2g_2}$ positive at any point $(L;L')\in H^+_{k+2g_1,l+2g_2}\cap U^\circ_{W,b,w}$.

Remark 3.7 (Definition of the subspaces $U_{W,b,w}$)**.** *The subspaces* $U_{W,b,w}$ *in Theorem 3.6 are defined as follows.*

Let $\mathbb{R}^k\times\mathbb{R}^l$ *have coordinates* $L_1,\dots,L_k;L'_1,\dots,L'_l$ *and let* $\mathbb{R}^{k+2g_1}\times\mathbb{R}^{l+2g_2}$ *have coordinates* $x_1,\dots,x_{k+2g_1};y_1,\dots,y_{l+2g_2}$. *Then the subspace* $U_{W,b,w}\subset\mathbb{R}^{k+2g_1}\times\mathbb{R}^{l+2g_2}$ *is defined as the preimage of the subspace* $W\subset\mathbb{R}^k\times\mathbb{R}^l$ *by the linear map:*

$$(x_1,\dots,x_{k+2g_1};y_1,\dots,y_{l+2g_2})\mapsto(L_1,\dots,L_k;L'_1,\dots,L'_l)$$

$$\begin{cases} L_1 = x_1+\ldots+x_{2b_1+1},\\ L_2 = x_{2b_1+2}+\ldots+x_{2b_1+2b_2+2},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ L'_1 = y_1+\ldots+y_{2w_1+1},\\ L'_2 = y_{2w_1+2}+\ldots+y_{2w_1+2w_2+2},\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

For example, if W is defined by the equation $L_1=L'_1$, then $U_{W,b,w}$ is defined by the equation $x_1+\ldots+x_{2b_1+1}=y_1+\ldots+y_{2w_1+1}$.

3.2.3 Chapuy-Féray-Fusy bijection

The main ingredient of the proof of Theorem 3.6 is a result from the theory of combinatorial maps, the Chapuy-Féray-Fusy bijection, which we now introduce.

In [Cha11] Chapuy introduced an operation ("slicing") on ribbon graphs with one boundary component which decreases the genus. Roughly speaking, the idea is as follows. In planar trees (genus 0 ribbon graphs with one boundary component), when we go along the unique boundary, we visit the corners adjacent to any fixed vertex in the same order as they are situated around that vertex. However, for the higher genus ribbon graphs this is no longer true in general. Such violation of order at a vertex allows us to "slice" this vertex into 3 new ones, preserving all edges and decreasing the genus of the ribbon graph by 1.

By iterating the slicing operation, one can obtain a genus 0 ribbon graph, i.e. a plane tree. To get back the initial ribbon graph, one has to "glue" some of the vertices of the tree. This suggests that there should be a bijection between ribbon graphs with one boundary component and plane trees with some decoration of their vertices. The complication is that at each step of slicing the choice of a vertex to slice is not canonical. Nevertheless, in [CFF13] Chapuy, Féray and Fusy gave such (non-explicit) bijection, which we will now present. This bijection also applies to vertex-labeled vertex-bicolored ribbon graphs ([CFF13, Section 3.3]), so we state right away the version for these graphs.

Recall from section 1.1.1 that a vertex-bicolored ribbon graph is *rooted* if it has a distinguished black corner, denoted by an oriented half edge pointing to it. Recall also that for $g \geq 0$, $k, l \geq 1$, $\mathcal{E}^{*,root}_{g,k,l}$ denotes the set of rooted vertex-bicolored ribbon graphs of genus g, with 1 boundary component, k black and l white labeled vertices.

Let $k, l \geq 1$. A *C-decorated* (k,l)*-tree* is a triple (T, σ_b, σ_w), where T is a rooted vertex-bicolored plane tree with k black and l white *non-labeled* vertices and σ_b, σ_w are permutations of the sets of black and white vertices of T respectively, such that:

- all cycles of σ_b and σ_w have odd length;
- each cycle carries a sign, either $+$ or $-$;
- the cycles of σ_b (respectively σ_w) are labeled from 1 to $|\sigma_b|$ (respectively $|\sigma_w|$), where $|\sigma|$ denotes the number of cycles in a permutation σ.

For $k, l, m, n \geq 1$, let $\mathcal{CT}_{k,l,m,n}$ be the set of C-decorated (k,l)-trees such that $|\sigma_b| = m$ and $|\sigma_w| = n$.

For a finite set $\mathcal{A}$ we denote by $n\mathcal{A}$ the set made of n disjoint copies of $\mathcal{A}$.

Theorem 3.8 (Chapuy, Féray, Fusy)**.** *For all $g \geq 0, k, l \geq 1$ there is a bijection*

$$2^{k+l+2g}\mathcal{E}^{*,root}_{g,k,l} \simeq \bigsqcup_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} \mathcal{CT}_{k+2g_1,l+2g_2,k,l},$$

such that the underlying graph of each ribbon graph can be obtained from the corresponding tree by merging into a single vertex the vertices in each cycle of σ_b and σ_w. The label of each cycle coincides with the label of the corresponding vertex.

We now give an informal explanation of how the features of this bijection allow us to obtain Theorem 3.6. The formal proof is given in the following subsection 3.2.4.

Firstly, note that, given a ribbon graph G of genus g and one of the C-decorated trees T corresponding to it via the bijection, the relation between the ribbon graph structures of G and T is not explicit. In other words, we know which vertices of T should be glued together to form G, but we do not know which corners of these vertices and in which cyclic order they should be glued.

However, we are only interested in the metrics on G. And the metric on a ribbon graph is only a property of the underlying graph (without the ribbon graph structure). So the non-explicit nature of the bijection does not pose a problem.

Moreover, the fact that the vertices of G are gluings of several vertices of T is very well suited to our problem. Indeed, we would like to count metrics on G with given vertex perimeters. The vertex perimeters of G are just the sums of vertex perimeters of the corresponding vertices of T. Since we are able to count metrics on trees with given vertex perimeters (subsection 3.2.1), this allows us to count the corresponding metrics on G via summation. For example, if a black vertex of G with perimeter L_1 is glued from 3 black vertices of T with perimeters x_1, x_2, x_3, etc., the count of metrics on G will be the sum over $\{(x_1, x_2, x_3) : x_1+x_2+x_3 = L_1\}$, etc., of the count of metrics on T.

The linear relations between the vertex perimeters of G (determined by a subspace $W \in \overline{\mathcal{W}_{k,l}}$) give rise in this way to linear relations between the vertex perimeters of T. This explains the appearance of the subspaces $U_{W,b,w}$ (Remark 3.7) and the corresponding counts of metrics $P^0_{U_{W,b,w}}$ in Theorem 3.6.

Finally, after performing the summation over all G, it turns out that the polynomiality of the top-degree term of $\mathcal{P}^g_{k,l}$ for $g > 0$ is the consequence of the polynomiality (constancy) for $g = 0$, which we have already proved in subsection 3.2.1.

3.2.4 Proof of Theorem 3.6

Let $g_1, g_2 \geq 0$ be such that $g_1 + g_2 = g$ and let $b = (b_1, \ldots, b_k)$, $w = (w_1, \ldots, w_l)$ be tuples of non-negative integers such that $b_1 + \ldots + b_k =$

$g_1, w_1 + \ldots + w_l = g_2$. Denote by $\mathcal{CT}_{k+2g_1,l+2g_2,k,l}(b,w)$ the subset of C-decorated trees $(T,\sigma_b,\sigma_w) \in \mathcal{CT}_{k+2g_1,l+2g_2,k,l}$ such that the (labeled) cycles of σ_b and σ_w have sizes $2b_1+1,\ldots,2b_k+1$ and $2w_1+1,\ldots,2w_l+1$ respectively.

Denote also by $\mathcal{E}^{*,root}_{g,k,l}(b,w)$ the subset of $2^{k+l+2g}\mathcal{E}^{*,root}_{g,k,l}$ which corresponds to $\mathcal{CT}_{k+2g_1,l+2g_2,k,l}(b,w)$ via the bijection of Theorem 3.8, so that

$$2^{k+l+2g}\mathcal{E}^{*,root}_{g,k,l} = \bigsqcup_{\substack{g_1+g_2=g\\ g_1,g_2\geq 0}} \bigsqcup_{\substack{b_1+\ldots b_k=g_1\\ w_1+\ldots+w_l=g_2}} \mathcal{E}^{*,root}_{g,k,l}(b,w).$$

Lemma 3.9. *There is a bijection*

$$\prod_{i=1}^{k}(2b_i+1)\prod_{j=1}^{l}(2w_j+1)\mathcal{E}^{*,root}_{g,k,l}(b,w) \simeq 2^{k+l}\mathcal{E}^{*,root}_{0,k+2g_1,l+2g_2}.$$

In addition, the underlying graph of the ribbon graph can be obtained from the corresponding tree by merging into a single vertex:

- *for each* $i=1,\ldots,k$*, black vertices with labels* $\sum_{r=1}^{i-1}(2b_r+1)+1,\ldots,\sum_{r=1}^{i}(2b_r+1)$ *to get the black vertex of* G *with label* i*;*
- *for each* $j=1,\ldots,l$*, white vertices with labels* $\sum_{r=1}^{j-1}(2w_r+1)+1,\ldots,\sum_{r=1}^{j}(2w_r+1)$ *to get the white vertex of* G *with label* j*.*

Proof. Consider $G \in \mathcal{E}^{*,root}_{g,k,l}(b,w)$. Let $(T,\sigma_b,\sigma_w) \in \mathcal{CT}_{k+2g_1,l+2g_2,k,l}(b,w)$ be the corresponding C-decorated tree. One can associate to T a family of rooted labeled trees from $\mathcal{E}^{*,root}_{0,k+2g_1,l+2g_2}$ by labelling the vertices of T in such a way that the first cycle of σ_b is $(1,2,\ldots,2b_1+1)$, the second cycle is $(2b_1+2,\ldots,2b_1+2b_2+2)$, etc., and similarly for σ_w; then forgetting both the signs of cycles of σ_b, σ_w and the permutations themselves (see Figure 3.4). This can be done in $\prod_{i=1}^{k}(2b_i+1)\prod_{j=1}^{l}(2w_j+1)$ ways, since it is enough to choose in each cycle the vertex which will have the minimal label.

All the rooted labeled trees we get by the procedure described above will actually constitute the whole set $\mathcal{E}^{*,root}_{0,k+2g_1,l+2g_2}$ and each tree $T' \in \mathcal{E}^{*,root}_{0,k+2g_1,l+2g_2}$ will be obtained 2^{k+l} times, because to recover a C-decorated tree from T' one firstly recovers the cycles of σ_b, σ_w and their labels from the labels of the vertices of T', but then one has to choose the signs of $k+l$ cycles of σ_b and σ_w. Then one recovers $G \in \mathcal{E}^{*,root}_{g,k,l}(b,w)$ via the bijection of Theorem 3.8. This gives the desired bijection. The statement about the underlying graph of G follows from the construction and from Theorem 3.8. $\square$

We are now ready to prove Theorem 3.6.

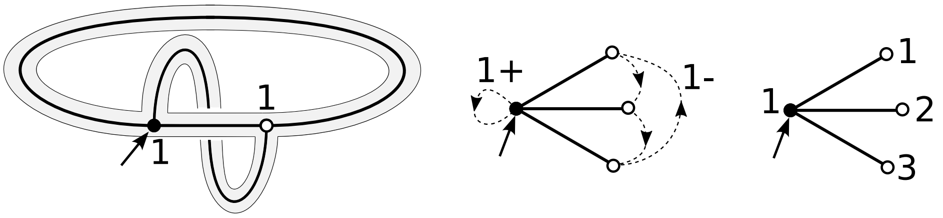


Figure 3.4: A ribbon graph from $\mathcal{E}^{*,root}_{1,1,1}$, one of its corresponding C-decorated trees from $\mathcal{CT}_{1,3,1,1}$, and one of the rooted labeled trees from $\mathcal{E}^{*,root}_{0,1,3}$ corresponding to this C-decorated tree.

Proof of Theorem 3.6. Fix $W \in \overline{\mathcal{W}_{k,l}}$. It follows from the first part of Lemma 2.22 that, for every connected component C of $H^+_{k,l} \cap W^\circ$ and for $(L, L') \in C$ we have

$$top_C\left(\mathcal{P}^g_{k,l}\right)(L, L') = \sum_{G \in \mathcal{E}^*_{g,k,l}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \left(\prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0}\right) \cdot \operatorname{Vol} M_G(L, L'), \tag{3.6}$$

where the polytope of metrics $M_G(L; L')$ is defined in (2.6).

We will show that these top-degree terms for different connected components coincide by giving an explicit expression for them, which will not depend on the connected component.

Any $G \in \mathcal{E}^*_{g,k,l}$ can be rooted at any of its $(k + l + 2g - 1)$ black corners. However, due to automorphisms, this makes $\frac{(k+l+2g-1)}{|\operatorname{Aut}(G)|}$ different rooted ribbon graphs. Hence (3.6) is equal to

$$(k + l + 2g - 1)^{-1} \cdot \sum_{G \in \mathcal{E}^{*,root}_{g,k,l}} \prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \operatorname{Vol} M_G(L, L'). \tag{3.7}$$

Fix $g_1, g_2 \geq 0$ with $g_1 + g_2 = g$ and $b = (b_1, \ldots, b_k)$, $w = (w_1, \ldots, w_l)$ tuples of non-negative integers such that $b_1 + \ldots + b_k = g_1, w_1 + \ldots + w_l = g_2$. Let $G \in \mathcal{E}^{*,root}_{g,k,l}(b, w)$ and let $T \in \mathcal{E}^{*,root}_{0,k+2g_1,l+2g_2}$ be one of the rooted labeled trees corresponding to G via the bijection of Lemma 3.9.

From Lemma 3.9 we know that the underlying graph of G can be obtained from T by merging its vertices in the corresponding groups. It means that one can choose an identification of the edges of G with the edges of T in such a way that an edge in T joining vertices from two groups is identified with an edge of G joining vertices that were merged from these two groups. Choose any such identification and, for each $e \in E(G)$, let $\widehat{e} \in E(T)$ be the corresponding edge in T.

$$
\begin{array}{ccc}
\mathbb{R}^{E(G)} & \xrightarrow{\ F\ } & \mathbb{R}^{E(T)} \\
\Big\downarrow \mathrm{vp}_G & & \Big\downarrow \mathrm{vp}_T \\
\mathbb{R}^k \times \mathbb{R}^l & \longleftarrow & \mathbb{R}^{k+2g_1} \times \mathbb{R}^{l+2g_2}
\end{array}
\tag{3.8}
$$

Consider the diagram (3.8). The identification of edges gives a linear isomorphism $F : \mathbb{R}^{E(G)} \to \mathbb{R}^{E(T)}$ between the spaces of weight functions on G and T. Let $w_e, e \in E(G)$ and $w_{\widehat{e}}, e \in E(G)$ be the standard coordinates on these spaces. Let also L, L' and x, y be the coordinates on the spaces of vertex perimeters $\mathbb{R}^k \times \mathbb{R}^l$ and $\mathbb{R}^{k+2g_1} \times \mathbb{R}^{l+2g_2}$ respectively. Note that by Lemma 2.5, since T is a tree, vp_T is an isomorphism. The composition $\mathrm{vp}_G \circ F^{-1} \circ \mathrm{vp}_T^{-1}$ is given by

$$
\begin{cases}
L_1 = x_1 + \ldots + x_{2b_1+1}, \\
L_2 = x_{2b_1+2} + \ldots + x_{2b_1+2b_2+2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
L'_1 = y_1 + \ldots + y_{2w_1+1}, \\
L'_2 = y_{2w_1+2} + \ldots + y_{2w_1+2w_2+2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{cases}
\tag{3.9}
$$

which follows from the construction of G by merging the vertices of T.

Denote $V(L, L') = \mathrm{vp}_T \circ F \circ \mathrm{vp}_G^{-1}(L, L')$. It is an affine subspace of $\mathbb{R}^{k+2g_1} \times \mathbb{R}^{l+2g_2}$ defined by equations (3.9). We claim that

$$
\prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \mathrm{Vol}\, M_G(L, L') = \int_{V(L,L')} \prod_{e \in E(T)} \mathbf{1}_{f_e(x,y)>0}\, d\,\mathrm{Vol}(x, y). \tag{3.10}
$$

Indeed, by the definition of $M_G(L, L')$ the left-hand side of (3.10) is equal to

$$
\prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \mathrm{Vol}\left[\mathrm{vp}_G^{-1}(L, L') \cap \{w_e > 0, e \in E(G) \setminus B(G)\}\right].
$$

Since both vp_T and F are invertible integral linear transformations, they both preserve integer lattices on affine subspaces, hence also the corresponding volumes. Hence we can apply $\mathrm{vp}_T \circ F$ to the expression inside Vol:

$$
\prod_{e \in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \mathrm{Vol}\left[\mathrm{vp}_T \circ F(\mathrm{vp}_G^{-1}(L, L')) \cap \mathrm{vp}_T \circ F(\{w_e > 0, e \in E(G) \setminus B(G)\})\right].
$$

Recall that $\mathrm{vp}_T \circ F \circ \mathrm{vp}_G^{-1}(L, L') = V(L, L')$ by definition. In addition, $w_e > 0 \iff w_{\widehat{e}} > 0$, and by Lemma 2.4 the edge weights in T are uniquely determined by vertex perimeters: $w_{\widehat{e}} = f_{\widehat{e}}(x,y)$. Hence the last expression is equal to

$$\prod_{e\in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \mathrm{Vol}\Big[V(L,L') \cap \{f_{\widehat{e}}(x,y) > 0, e \in E(G)\setminus B(G)\}\Big]$$
$$= \prod_{e\in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \int_{V(L,L')} \prod_{e\in E(G)\setminus B(G)} \mathbf{1}_{f_{\widehat{e}}(x,y)>0}\, d\,\mathrm{Vol}(x,y)$$

For $(x,y) \in V(L,L')$ and $e \in B(G)$ we have $f_e(L,L') = w_e = w_{\widehat{e}} = f_{\widehat{e}}(x,y)$, so we finally get

$$\int_{V(L,L')} \prod_{e\in E(G)} \mathbf{1}_{f_{\widehat{e}}(x,y)>0}\, d\,\mathrm{Vol}(x,y) = \int_{V(L,L')} \prod_{e\in E(T)} \mathbf{1}_{f_e(x,y)>0}\, d\,\mathrm{Vol}(x,y),$$

which is the right-hand side of (3.10).

Now, summing the equality (3.10) over all $G \in \prod_{i=1}^k(2b_i+1)\prod_{j=1}^l(2w_j+1)\mathcal{E}^{*,root}_{g,k,l}(b,w)$ and applying Lemma 3.9 we get

$$\prod_{i=1}^k(2b_i+1)\prod_{j=1}^l(2w_j+1) \sum_{G\in\mathcal{E}^{*,root}_{g,k,l}(b,w)} \prod_{e\in B(G)} \mathbf{1}_{f_e(L,L')>0} \cdot \mathrm{Vol}\, M_G(L,L')$$
$$= 2^{k+l} \int_{V(L,L')} \left(\sum_{T\in\mathcal{E}^{*,root}_{0,k+2g_1,l+2g_2}} \prod_{e\in E(T)} \mathbf{1}_{f_e(x,y)>0} \right) d\,\mathrm{Vol}(x,y).$$

Since each tree $T \in \mathcal{E}^*_{0,k+2g_1,l+2g_2}$ can be rooted at any of its $(k+l+2g-1)$ black corners and T has no non-trivial automorphisms (as noted at the beginning of Section 3.2.1), the last expression is equal to

$$2^{k+l}(k+l+2g-1) \int_{V(L,L')} \left(\sum_{T\in\mathcal{E}^*_{0,k+2g_1,l+2g_2}} \prod_{e\in E(T)} \mathbf{1}_{f_e(x,y)>0} \right) d\,\mathrm{Vol}(x,y).$$

Note that from (3.9) it follows that when $(L,L') \in H^+_{k,l} \cap W^\circ$, the generic point (x,y) of the affine subspace $V(L,L')$ lies in $(U_{W,b,w})^\circ$ for some corresponding $U_{W,b,w} \in \overline{\mathcal{W}_{k+2g_1,l+2g_2}}$, defined in Remark 3.7.

Consider the sum inside the integral in the last expression. It was proven in Section 3.2.1 that this sum is constant on $H^+_{k+2g_1,l+2g_2} \cap (U_{W,b,w})^\circ$ with the corresponding value $P^0_{U_{W,b,w}}$, and is zero outside of $H^+_{k+2g_1,l+2g_2}$. So the last expression is equal to

$$\begin{aligned}
&2^{k+l}(k+l+2g-1)\int_{V(L,L')\cap H^+_{k+2g_1,l+2g_2}} P^0_{U_{W,b,w}}\, d\,\mathrm{Vol}(x,y)\\
&= 2^{k+l}(k+l+2g-1)\cdot P^0_{U_{W,b,w}}\cdot \mathrm{Vol}(V(L,L')\cap H^+_{k+2g_1,l+2g_2})\\
&= 2^{k+l}(k+l+2g-1)\cdot P^0_{U_{W,b,w}}\cdot \prod_{i=1}^{k}\frac{L_i^{2b_i}}{(2b_i)!}\cdot\prod_{j=1}^{l}\frac{{L'_j}^{2w_j}}{(2w_j)!}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{G\in\mathcal{E}^{*,root}_{g,k,l}(b,w)}\ \prod_{e\in B(G)} \mathbf{1}_{f_e(L,L')>0}\cdot \mathrm{Vol}\, M_G(L,L')\\
&= 2^{k+l}(k+l+2g-1)\cdot P^0_{U_{W,b,w}}\cdot \prod_{i=1}^{k}\frac{L_i^{2b_i}}{(2b_i+1)!}\cdot\prod_{j=1}^{l}\frac{{L'_j}^{2w_j}}{(2w_j+1)!}.
\end{aligned}$$

Summing this over all g_1, g_2 and all b, w we get

$$\begin{aligned}
&2^{k+l+2g}\sum_{G\in\mathcal{E}^{*,root}_{g,k,l}}\ \prod_{e\in B(G)} \mathbf{1}_{f_e(L,L')>0}\cdot \mathrm{Vol}\, M_G(L,L')\\
&= 2^{k+l}(k+l+2g-1)\cdot \sum_{\substack{b_1+\ldots+b_k+w_1+\ldots+w_l=g\\ b_i,w_i\geq 0}} P^0_{U_{W,b,w}}\cdot \prod_{i=1}^{k}\frac{L_i^{2b_i}}{(2b_i+1)!}\cdot\prod_{j=1}^{l}\frac{{L'_j}^{2w_j}}{(2w_j+1)!}.
\end{aligned}$$

Taking (3.7) into account we finally get the expression (3.5) which does not depend on the connected component C of $H^+_{k,l}\cap W^\circ$. □

3.3 Explicit top-degree terms on $H_{k,l}$ and V_n

Recall the definitions of the following subspaces:

$$\begin{aligned}
H_{k,l} &= \{L_1+\ldots+L_k = L'_1+\ldots+L'_l\}\subset\mathbb{R}^k\times\mathbb{R}^l,\\
V_n &= \{L_1 = L'_1,\ldots,L_n = L'_n\}\subset\mathbb{R}^n\times\mathbb{R}^n.
\end{aligned}$$

In this section we find explicit expressions for $P^g_{H_{k,l}}$ and $P^g_{V_n}$ as polynomials in L, L'.

The starting point is Theorem 3.6 proven in the previous section. It states that for any g and W there is a polynomial expression for P^g_W whose coefficients are the constants $P^0_{U_{W,b,w}}$ for some particular subspaces $U_{W,b,w}$

as defined in Remark 3.7. It is thus enough to compute these constants (explicitly or recursively).

To this end, we introduce the following notations (to be used only in this section). Let $n, k, l \geq 1$ and let $b_1 + \ldots + b_n = k$, $w_1 + \ldots + w_n = l$, with b_i, w_i positive integers. Denote by $W^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$ the subspace in $\overline{\mathcal{W}_{k,l}}$ defined by the equations

$$\begin{aligned} L_1 + \ldots + L_{b_1} &= L'_1 + \ldots + L'_{w_1}, \\ L_{b_1+1} + \ldots + L_{b_1+b_2} &= L'_{w_1+1} + \ldots + L'_{w_1+w_2}, \\ &\ldots \\ L_{b_1+\ldots+b_{n-1}+1} + \ldots + L_k &= L'_{w_1+\ldots+w_{n-1}+1} + \ldots + L'_l. \end{aligned} \tag{3.11}$$

For example, W^k_l is simply the ambient subspace $H_{k,l}$.

Denote by $p^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$ the unique value of $P^0_{W^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}}$.

As explained in section 3.2.1, $p^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$ is equal to the number of trees $G \in \mathcal{E}^*_{0,k,l}$ positive at (L, L') for any $(L, L') \in H^+_{k,l} \cap (W^{b_1,\ldots,b_n}_{w_1,\ldots,w_n})^\circ$. We will use this combinatorial interpretation of these numbers throughout this section.

3.3.1 Proof of Proposition 3.3

Lemma 3.10. *For all $k, l \geq 1$ we have $p^k_l = (k + l - 2)!$.*

Proof. Consider the point $(L, L') = (N, 1, \ldots, 1; \frac{N+k-1}{l}, \ldots, \frac{N+k-1}{l}) \in \mathbb{R}^k \times \mathbb{R}^l$ with $N \gg kl$. It is easy to check that it belongs to $H^+_{k,l} \cap (W^k_l)^\circ = H^+_{k,l} \cap (H_{k,l})^\circ$. Thus p^k_l is the number of trees positive at (L, L'). These vertex perimeters force a particularly simple structure of the positive trees. Since $\frac{N+k-1}{l} \cdot (l-1) < N$, all of the l white vertices must be adjacent to the black vertex with perimeter N. The remaining $k-1$ black vertices with perimeters 1 can be attached to the white vertices in an arbitrary manner. The total number of positive trees is then

$$(l-1)! \cdot l \cdot (l+1) \cdot \ldots \cdot (l+k-2) = (k+l-2)!.$$

□

Proof of Proposition 3.3. Fix g, k, l and let $b = (b_1, \ldots, b_k)$ and $w = (w_1, \ldots, w_l)$ be vectors of non-negative integers with $b_1 + \ldots + b_k = g_1$, $w_1 + \ldots + w_l = g_2$ and $g_1 + g_2 = g$.

Remark 3.7 implies that

$$U_{H_{k,l},b,w} = W^{k+2g_1}_{l+2g_2} = H_{k+2g_1,l+2g_2} \subset \mathbb{R}^{k+2g_1} \times \mathbb{R}^{l+2g_2}.$$

Thus, by Lemma 3.10, $P^0_{U_{H_{k,l},b,w}} = p^{k+2g_1}_{l+2g_2} = (k+l+2g-2)!$, irrespective of the vectors b and w. Plugging this value in the expression for $P^g_{H_{k,l}}$ given by Theorem 3.6, we get the desired formula. □

3.3.2 Proof of Theorem 3.4

Using the notations introduced in this section, we can rewrite the expression for $P^g_{V_n}$ given by Theorem 3.6 and Remark 3.7 as

$$P^g_{V_n}(L,L) = 2^{-2g} \cdot \sum_{\substack{b_1+\ldots+b_n+w_1+\ldots+w_n=g \\ b_i,w_i\geq 0}} p^{2b_1+1,\ldots,2b_n+1}_{2w_1+1,\ldots,2w_n+1} \cdot \prod_{i=1}^{n} \frac{L_i^{2(b_i+w_i)}}{(2b_i+1)!(2w_i+1)!}. \tag{3.12}$$

We start by giving a recurrence relation for the numbers $p^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$. It will be obtained by implementing the general strategy for the computation of top-degree terms of the counting functions on the cells of $\mathcal{PS}_{k,l}$ of positive codimension, described in section 2.2.5. In the notations of that section, the families $\mathcal{G}_{g,(k,l)}$ that we consider are the families of plane trees $\mathcal{E}^*_{0,k,l}$. We will thus study degenerations of plane trees.

Proposition 3.11. *Let $n,k,l \geq 1$ and let $b_i, w_i \geq 1$ such that $b_1+\ldots+b_n = k$ and $w_1+\ldots+w_n = l$. Then*

$$(k+l-2)! = p^{b_1,\ldots,b_n}_{w_1,\ldots,w_n} + \sum_{t=2}^{n} \frac{(k+l-2)_{t-2}}{t!} \sum_{I_1,\ldots I_t} \prod_{j=1}^{t} \left(\sum_{i\in I_j} (b_i+w_i) - 1 \right) p^{b_{I_j}}_{w_{I_j}}, \tag{3.13}$$

where $(x)_t := x(x-1)\cdots(x-t+1)$ is the falling factorial ($(x)_0 := 1$); the second sum is over all partitions of $\{1,\ldots,n\}$ into t non-empty labeled sets $I_1,\ldots,I_t$; b_{I_j} denotes $\{b_i\}_{i\in I_j}$, and analogously for w_{I_j}.

Proof. Let $(L_1,\ldots,L_k; L'_1,\ldots,L'_l)$ be a point in $H^+_{k,l} \cap (W^{b_1,\ldots,b_n}_{w_1,\ldots,w_n})^\circ$. For $i = 1,\ldots,n$, let $A_{b,i} = \{b_1+\ldots+b_{i-1}+1,\ldots,b_1+\ldots+b_i\}$ and $A_{w,i} = \{w_1+\ldots+w_{i-1}+1,\ldots,w_1+\ldots+w_i\}$. The defining equations (3.11) of the wall $W^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$ can now be rewritten as

$$\sum_{j\in A_{b,i}} L_j = \sum_{j\in A_{w,i}} L'_j, \; i = 1,\ldots,n. \tag{3.14}$$

Consider now a path of the form

$$(L_1+(k-1)\varepsilon, L_2-\varepsilon,\ldots,L_k-\varepsilon; L'_1,\ldots,L'_k), \; \varepsilon \to 0, \varepsilon > 0.$$

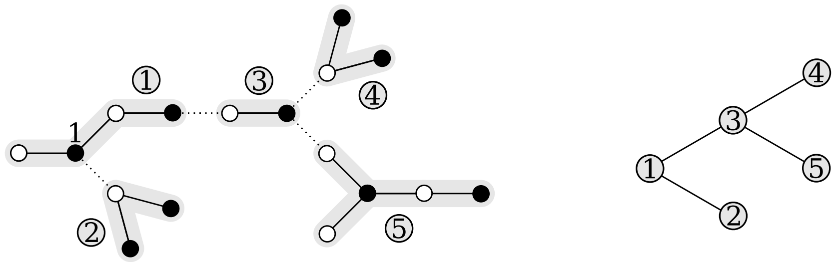


Figure 3.5: Left: a degenerate tree $G \in \mathcal{D}$ with $t = 5$ constituent positive trees, which are shaded in grey. Their labels are circled. Zero-weight edges are dotted. Black vertex number 1 is in the positive tree number 1. Note that for every zero-weight edge e in G, the black extremity of e is in the same connected component of $G - e$ as the black vertex number 1. Right: the corresponding tree T.

While $\varepsilon > 0$ and ε is sufficiently small, the point in question lies in $H^+_{k,l} \cap (H_{k,l})^\circ$ and the number of trees positive at this point is equal to $(k + l - 2)!$ by Lemma 3.10. When $\varepsilon = 0$, the number of trees positive at this point is equal to $p^{b_1,\dots,b_n}_{w_1,\dots,w_n}$ by definition. Hence we can deduce the desired formula (3.13) if we enumerate the trees that cease to be positive when $\varepsilon \to 0$.

Let $\mathcal{D}$ be the set of these degenerate trees, and consider a tree from $\mathcal{D}$. It consists of several positive trees connected by zero-weight edges (Figure 3.5, left). Applying the edge weight formula of Lemma 2.4 to a zero-weight edge, we obtain a linear relation between the vertex perimeters of the form $\sum_{i \in A_b} L_i = \sum_{j \in A_w} L'_j$. Since $(L_1, \dots, L_k; L'_1, \dots, L'_l) \in (W^{b_1,\dots,b_n}_{w_1,\dots,w_n})^\circ$, the only such linear relations possible are given by the equations (3.14) and the sums of such equations. Hence $A_b = \bigcup_{i \in I} A_{b,i}$, $A_w = \bigcup_{i \in I} A_{w,i}$ for some index set $I \subset \{1, \dots, n\}$. This implies the following condition on the trees in $\mathcal{D}$.

Condition 1. In every constituent positive tree the set of vertex labels is of the form $\bigcup_{i \in I} A_{b,i}$ for black vertices and $\bigcup_{i \in I} A_{w,i}$ for white vertices, for some index set I.

Consider a zero-weight edge e of a degenerate tree G from $\mathcal{D}$. We claim that the black extremity of e is in the same connected component of $G - e$ as the black vertex number 1. Indeed, if it were not true, then by Lemma 2.4 the weight of this edge when $\varepsilon > 0$ would be equal to

$$\sum_{i \in I} \left(\sum_{j \in A_{b,i}} (L_j - \varepsilon) - \sum_{j \in A_{w,i}} L'_j \right) = -\left(\sum_{i \in I} b_i \right) \cdot \varepsilon < 0,$$

for some index set $I \subset \{1, \dots, n\}$, a contradiction. The above argument implies two more conditions on trees in $\mathcal{D}$.

Condition 2. The zero-weight edges that are incident to the positive tree containing the black vertex number 1 are incident only to its black vertices.

Condition 3. For each positive tree not containing the black vertex number 1 there is exactly one incident zero-weight edge which is incident to its white vertex, and several (maybe none) incident zero-weight edges which are incident to its black vertices.

We now claim that, conversely, Conditions 1, 2 and 3 completely characterize trees in $\mathcal{D}$. More precisely, if one:

- chooses a partition $I_1, \dots, I_t$ of $\{1, \dots, n\}$ into $t \geq 2$ non-empty sets;
- for each j chooses a tree on the black vertices with labels $\bigcup_{i \in I_j} A_{b,i}$ and the white vertices with labels $\bigcup_{i \in I_j} A_{w,i}$, which is positive given the corresponding vertex perimeters L_s, $s \in \bigcup_{i \in I_j} A_{b,i}$ and L'_s, $s \in \bigcup_{i \in I_j} A_{w,i}$;
- joins these trees by edges respecting Conditions 2 and 3, to form a single tree;

then one gets a tree in $\mathcal{D}$. Indeed, Lemma 2.4 and Condition 1 assure that, in the resulting tree, the joining edges have length zero at the point $(L_1, \dots, L_k; L'_1, \dots, L'_l)$, while Lemma 2.4 and Conditions 2 and 3 assure that their lengths are positive when $\varepsilon > 0$ and sufficiently small.

Having characterized trees in $\mathcal{D}$, we are ready to count them. In the desired formula (3.13): $2 \leq t \leq n$ stands for the number of constituent positive trees, which we label by numbers from 1 to t for convenience; the factor $1/t!$ accounts for the arbitrariness of the numbering of these trees; $I_1, \dots, I_t$ are such that the vertex labels of the j-th positive tree are $\bigcup_{i \in I_j} A_{b,i}$ and $\bigcup_{i \in I_j} A_{w,i}$; $\prod_{j=1}^{t} p_{w_{I_j}}^{b_{I_j}}$ is the number of possible choices of positive trees themselves. We now fix t, the partition $I_1, \dots, I_t$, and a choice of t positive trees, and we will count the number of ways to connect them with zero-weight edges to form a degenerate tree from $\mathcal{D}$ (i.e. respecting Conditions 2 and 3). We claim that this count gives the remaining factor

$$(k + l - 2)_{t-2} \cdot \prod_{j=1}^{t} \left(\sum_{i \in I_j} (b_i + w_i) - 1 \right). \tag{3.15}$$

Without loss of generality, assume that the black vertex number 1 is in the positive tree number 1 (the resulting count of degenerate trees will not depend on this choice). To each way of connecting the positive trees with zero-weight edges we put into correspondence a non-plane labeled tree T with t vertices, where the vertex number j of T corresponds to the positive tree number j, and the edges of T correspond to zero-weight edges joining the positive trees (Figure 3.5, right).

Let $c_j = \sum_{i\in I_j}(b_i + w_i)$ be the number of vertices in the positive tree number j. Consider a tree T such that the degree of the vertex number j is equal to d_j. It follows from Conditions 2 and 3 that there are

$$(c_1-1)^{(d_1)}\cdot\prod_{j=2}^{t}(c_j-1)(c_j-1)^{(d_j-1)}$$

ways of connecting the positive trees with zero-weight edges to form a degenerate tree from $\mathcal{D}$ which corresponds to T; here $x^{(n)} = x(x+1)\cdots(x+n-1)$ is the rising factorial ($x^{(0)} := 1$). Indeed, a bipartite plane tree with N vertices has $N-1$ corners around black (white) vertices where we can glue an edge. If we have glued an edge to a black (white) vertex, the number of corners available for gluing around black (white) vertices increases by one. This gives the rising factorials in the formula.

It is well known that the number of non-plane labeled trees T on t vertices with the degree of the vertex number i equal to d_i is

$$\binom{t-2}{d_1-1,\dots,d_t-1}$$

provided $d_1+\ldots+d_t = 2t-2$. Hence the missing factor in our desired formula is equal to

$$\begin{aligned}
&\prod_{j=1}^{t}(c_j-1)\cdot\sum_{\substack{d_1+\ldots+d_t=2t-2\\ d_i\geq 1}}\binom{t-2}{d_1-1,\dots,d_t-1}c_1^{(d_1-1)}\prod_{j=2}^{t}(c_j-1)^{(d_j-1)}\\
&=\prod_{j=1}^{t}(c_j-1)\cdot(t-2)!\cdot\sum_{\substack{d'_1+\ldots+d'_t=t-2\\ d'_i\geq 0}}\binom{c_1-1+d'_1}{d'_1}\prod_{j=2}^{t}\binom{c_j-2+d'_j}{d'_j}\\
&=\prod_{j=1}^{t}(c_j-1)\cdot(t-2)!\cdot\binom{k+l-2}{t-2}=\prod_{j=1}^{t}(c_j-1)\cdot(k+l-2)_{t-2}.
\end{aligned}$$

In the first equality we perform the change of variables $d'_i = d_i-1$. The second equality follows from the following general identity, valid for $n, m \in \mathbb{N}\cup\{0\}$, $x_1,\dots,x_n\in\mathbb{R}$:

$$\sum_{\substack{d_1+\ldots+d_n=m\\ d_i\in\mathbb{N}\cup\{0\}}}\binom{x_1+d_1}{d_1}\cdots\binom{x_n+d_n}{d_n}=\binom{\sum_{i=1}^{n}x_i+(n-1)+m}{m},$$

which can be obtained by extracting the coefficient of z^m on both sides of the identity

$$(1-z)^{-(x_1+1)}\cdots(1-z)^{-(x_n+1)} = (1-z)^{-(x_1+\ldots+x_n+(n-1)+1)}.$$

We also use the fact that $c_1+\ldots+c_t = k+l$. $\square$

Corollary 3.12. *The value of $p^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$ depends only on b_i+w_i, $i=1,\ldots,n$.*

Proof. The proof is by induction on n. The base case $n=1$ follows from the explicit formula of Lemma 3.10. The induction step follows from the formula of Proposition 3.11, because the left hand side is just $(\sum_i(b_i+w_i)-2)!$, and the big sum on the right hand side depends only on b_i+w_i by the induction hypothesis (the numbers $p^{b_{I_j}}_{w_{I_j}}$ have strictly less then n indices). $\square$

Proof of Theorem 3.4. Denote by $p_{s_1,\ldots,s_n}$ the common value of $p^{b_1,\ldots,b_n}_{w_1,\ldots,w_n}$ with $b_i+w_i=s_i$, which is well-defined by Corollary 3.12. Then it follows from (3.12) that

$$P^g_{V_n}(L,L) = 2^{-2g}\cdot\sum_{\substack{s_1+\ldots+s_n=g\\ s_i\geq 0}} p_{2s_1+2,\ldots,2s_n+2}\cdot\prod_{i=1}^n\left(\sum_{b_i+w_i=s_i}\frac{L_i^{2s_i}}{(2b_i+1)!(2w_i+1)!}\right)$$

$$= 2^{-2g}\cdot\sum_{\substack{s_1+\ldots+s_n=g\\ s_i\geq 0}} p_{2s_1+2,\ldots,2s_n+2}\cdot\prod_{i=1}^n\left(L_i^{2s_i}\cdot\frac{2^{2s_i+1}}{(2s_i+2)!}\right)$$

$$= 2^n\cdot\sum_{\substack{s_1+\ldots+s_n=g\\ s_i\geq 0}} p_{2s_1+2,\ldots,2s_n+2}\prod_{i=1}^n\frac{L_i^{2s_i}}{(2s_i+2)!}$$

$$= 2^n\cdot\sum_{\substack{s_1+\ldots+s_n=g\\ s_i\geq 1}} p_{2s_1,\ldots,2s_n}\prod_{i=1}^n\frac{L_i^{2s_i-2}}{(2s_i)!}.$$

In the second equality we have used the fact that $\sum_{b_i+w_i=s_i}\binom{2s_i+2}{2b_i+1} = 2^{2s_i+1}$ and so $\sum_{b_i+w_i=s_i}\frac{1}{(2b_i+1)!(2w_i+1)!} = \frac{2^{2s_i+1}}{(2s_i+2)!}$. The last equality is just a change of variables $s_i = s_i+1$.

We have obtained the desired expression for $P^g_{V_n}$. Now we have to show that the generating function $\mathcal{T}$ of the numbers $p_{s_1,\ldots,s_n}$ satisfies the relation (3.1).

The recurrence relation (3.13) can be rewritten with the new notation $s_i=b_i+w_i$, $s=s_1+\ldots+s_n$ as

$$(s-2)! = p_{s_1,\ldots,s_n} + \sum_{t=2}^n\frac{(s-2)_{t-2}}{t!}\cdot\sum_{I_1,\ldots I_t}\prod_{j=1}^t\left(\sum_{i\in I_j}s_i-1\right)\cdot p_{s_{I_j}}, \qquad (3.16)$$

where the second sum is over all partitions of $\{1,\ldots,n\}$ into t non-empty labeled sets $I_1,\ldots,I_t$, and s_{I_j} denotes $\{s_i\}_{i\in I_j}$.

Now multiply (3.16) by $s(s-1)\frac{1}{n!}t_{s_1}\cdots t_{s_n}$ and sum over all $n\geq 1$ and all $s_1,\ldots,s_n\geq 2$ such that $s_1+\ldots+s_n=s$. The left hand side becomes

$$s!\cdot\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{s_1+\ldots+s_n=s\\ s_i\geq 2}}t_{s_1}\cdots t_{s_n}=s!\cdot[t^s]\exp\left(\sum_{i\geq 2}t_it^i\right). \tag{3.17}$$

The right-hand side becomes

$$\sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{s_1+\ldots+s_n=s\\ s_i\geq 2}}\sum_{t=1}^{n}\binom{s}{t}\sum_{I_1,\ldots I_t}\prod_{j=1}^{t}\left(\sum_{i\in I_j}s_i-1\right)p_{s_{I_j}}\prod_{i\in I_j}t_{s_i}.$$

Changing the order of summation, this is equal to

$$\sum_{t=1}^{s}\binom{s}{t}\sum_{n\geq t}\frac{1}{n!}\sum_{\substack{s_1+\ldots+s_n=s\\ s_i\geq 2}}\sum_{I_1,\ldots I_t}\prod_{j=1}^{t}\left(\sum_{i\in I_j}s_i-1\right)p_{s_{I_j}}\prod_{i\in I_j}t_{s_i}.$$

Denote $\Sigma_j=\sum_{i\in I_j}s_i$ and $n_j=|I_j|$. Let also $I_j=\{i_1^j,\ldots,i_{n_j}^j\}$ with $i_1^j<\ldots<i_{n_j}^j$.

Fix s, t and n. To every pair consisting of a composition $s_1+\ldots+s_n=s$ and a labeled partition $I_1\sqcup\ldots\sqcup I_t=\{1,\ldots,n\}$ we put into correspondence a composition $\Sigma_1+\ldots+\Sigma_t=s$ and t compositions $s_{i_1^j}+\ldots+s_{i_{n_j}^j}=\Sigma_j$. This correspondence is $\frac{n!}{n_1!\cdots n_t!}$-to-1, because the tuple $(s_1,\ldots,s_n)$ can be uniquely reconstructed if the partition $I_1\sqcup\ldots\sqcup I_t$ is known, and there are $\frac{n!}{n_1!\cdots n_t!}$ ways to choose this partition. Hence the last sum can be rewritten as

$$\sum_{t=1}^{s}\binom{s}{t}\sum_{\Sigma_1+\ldots+\Sigma_t=s}\prod_{j=1}^{t}(\Sigma_j-1)\left(\sum_{n_j\geq 1}\frac{1}{n_j!}\sum_{s_{i_1^j}+\ldots+s_{i_{n_j}^j}=\Sigma_j}p_{s_{i_1^j},\ldots,s_{i_{n_j}^j}}t_{s_{i_1^j}}\cdots t_{i_{n_j}^j}\right).$$

This last expression is equal to

$$[t^s]\mathcal{T}(t,t_2,t_3,\ldots)^s. \tag{3.18}$$

Equating (3.17) and (3.18) we get relation (3.1) for $k\geq 2$. For $k=0,1$ this relation can be easily verified from definitions. $\square$

3.4 Cylinder contributions

To prove Theorem 3.5, we first have to relate the counting functions $\mathcal{P}^g_{k,l}$ and the counts of square-tiled surfaces in $\mathcal{H}(2g-2)$ with a fixed number of cylinders.

Proposition 3.13. *The number $|\mathcal{ST}_n(\mathcal{H}(2g-2), N)|$ of n-cylinder square-tiled surfaces in $\mathcal{H}(2g-2)$ with at most N squares is equal to*

$$\frac{1}{n!} \cdot \sum_{\substack{\sum_{i=1}^n h_i L_i \le N \\ h_i, L_i \in \mathbb{Z}_{>0}}} L_1 \cdots L_n \cdot \mathcal{P}^{g-n}_{n,n}(L_1, \ldots, L_n; L_1, \ldots, L_n). \tag{3.19}$$

Proof. Consider a n-cylinder square-tiled surface S in $\mathcal{H}(2g-2)$, with cylinders arbitrarily labeled from 1 to n. Recall from section 1.3.4 that we denote by G_S the union of all conical singularities and horizontal saddle connections of S. Clearly, G_S is a one-vertex ribbon graph (whose vertex is the unique conical singularity). Moreover, the boundary components of G_S come in two types, depending on whether the bottom or the top sides of the squares are glued to this boundary. We color the first boundaries in black and the second ones in white. Clearly, adjacent boundaries have opposite colors. Each of the n labeled cylinders of $S \setminus G_S$ is glued to one black and one white boundary of G_S, so G_S has n black and n white boundaries, which we label by the label of the adjacent cylinder. Finally, gluing a cylinder increases the genus of the surface by 1, so the genus of G_S must be equal to $g-n$. Hence, $G_S \in \mathcal{E}_{g-n,n,n}$.

We now prove the formula (3.19). The square-tiled cylinders of $S \setminus G_S$ are uniquely specified by their heights $h_i \in \mathbb{Z}_{>0}$ and their circumferences $L_i \in \mathbb{Z}_{>0}$, $1 \le i \le n$. The total number of squares in the surface is then $\sum_{i=1}^n h_i L_i$, which gives the inequality condition. Each edge of G_S is endowed with a positive integer length equal to the number of squares glued to either of its sides, and the perimeter of each boundary component of G_S is equal to the circumference of the adjacent cylinder. This gives the term $\mathcal{P}^{g-n}_{n,n}(L_1, \ldots, L_n; L_1, \ldots, L_n)$. The term $L_1 \cdots L_n$ comes from the fact that for the cylinder with label i, there are L_i different ways to twist it before gluing to G_S. Finally, the term $\frac{1}{n!}$ accounts for the arbitrariness of the numbering of the n cylinders. □

Remark 3.14. *Recall Lemma 1.11. Its proof can be found in the paper [AEZ14], Lemma 3.7. The proof proceeds by approximating the properly normalized initial sum by an integral of a polynomial over the standard simplex. In particular, the statement also holds when the function being summed is a polynomial in the variables $L_1, \ldots, L_n$ divisible by $L_1 \cdots L_n$, but only outside* *of a finite number of hyperplanes (measure zero set), where it is given by polynomials of at most the same degree (this last condition is sufficient to ensure that the term with the integral over the "exceptional" locus does not contribute to the asymptotics when $N \to \infty$).*

We can now give an explicit expression for the volume contributions $\mathrm{Vol}_n(2g-2)$ in terms of the numbers $p_{s_1,\ldots,s_n}$ defined in Theorem 3.4.

Proposition 3.15. *For $g \geq 1$ and $1 \leq n \leq g$:*

$$\mathrm{Vol}_n(2g-2) = \frac{2}{(2g-1)!} \cdot \frac{1}{n!} \cdot \sum_{\substack{s_1+\ldots+s_n=g \\ s_i \geq 1}} p_{2s_1,\ldots,2s_n} \frac{\zeta(2s_1)}{s_1} \cdots \frac{\zeta(2s_n)}{s_n},$$

where ζ is the Riemann zeta function.

Equivalently,

$$a_{g,n} = \frac{1}{n!} \cdot \sum_{\substack{s_1+\ldots+s_n=g \\ s_i \geq 1}} p_{2s_1,\ldots,2s_n} \cdot \prod_{i=1}^{n} \frac{(-1)^{s_i+1} B_{2s_i}}{2s_i \cdot (2s_i)!},$$

where B_i is the i-th Bernoulli number.

Proof. Combining the formula (3.2) and Proposition 3.13, we see that

$$\mathrm{Vol}_n(2g-2) = 2 \cdot 2g \cdot \frac{1}{n!} \cdot \lim_{N\to\infty} N^{-2g} \cdot \sum_{\substack{\sum_{i=1}^n h_i L_i \leq N \\ h_i, L_i \in \mathbb{Z}_{>0}}} L_1 \cdots L_n \cdot \mathcal{P}_{n,n}^{g-n}(L_1,\ldots,L_n,L_1,\ldots,L_n).$$

By Remark 3.14, we can replace $\mathcal{P}_{n,n}^{g-n}$ by its top-degree term $P_{V_n}^{g-n}$. Then, substituting the explicit formula for the polynomial $P_{V_n}^{g-n}$ from Theorem 3.4, and using Lemma 1.11 we get

$$\frac{2}{(2g-1)!} \cdot \frac{1}{n!} \cdot \sum_{\substack{s_1+\ldots+s_n=g \\ s_i \geq 1}} p_{2s_1,\ldots,2s_n} \frac{\zeta(2s_1)}{s_1} \cdots \frac{\zeta(2s_n)}{s_n},$$

which is the first formula of Proposition 3.15.

The second formula follows from the fact that $\mathrm{Vol}_n(2g-2) = \frac{2(2\pi)^{2g}}{(2g-1)!} a_{g,n}$ by definition, and that for $s \in \mathbb{Z}_{>0}$ one has $\zeta(2s) = \frac{(-1)^{s+1} B_{2s} (2\pi)^{2s}}{2 \cdot (2s)!}$, where B_{2s} is the $2s$-th Bernoulli number. ∎

Proof of Theorem 3.5. It follows from the equality $\mathrm{Vol}_n(2g-2) = \frac{2(2\pi)^{2g}}{(2g-1)!} a_{g,n}$ and from the explicit expression for $\mathrm{Vol}_n(2g-2)$ from Proposition 3.15 that $\mathcal{C}(2\pi t, u)$ is equal to $\mathcal{T}(t, t_2, t_3, \ldots)$ evaluated at $t_{2i} = \frac{\zeta(2i)}{i} u$ and $t_{2i+1} = 0$ for $i \geq 1$, where $\mathcal{T}$ is defined in Theorem 3.4. Then the relation (3.1) from Theorem 3.4 gives for $k = 2g, g \geq 0$.

$$\frac{1}{(2g)!}[t^{2g}]\mathcal{C}(2\pi t, u)^{2g} = [t^{2g}] \exp\left(u \cdot \sum_{i\geq 1} \frac{\zeta(2i)}{i} t^{2i}\right).$$

The series inside the exponent can be rewritten in terms of logarithms (as noted in Lemma 3.8 of [DGZZ22]): expanding the definition of the zeta function and changing the order of summation we find that it is equal to

$$u \cdot \sum_{i\geq 1} \log\left(1 - \frac{t^2}{i^2}\right),$$

so the exponent is

$$\left(\prod_{i\geq 1}\left(1 - \frac{t^2}{i^2}\right)\right)^u = \left(\frac{\sin(\pi t)}{\pi t}\right)^u,$$

where we have used the well-known product formula for the sine function. Replacing $2\pi t$ by t, we get the desired relation (3.3). □

Chapter 4

Spin parity of one-vertex graphs

In this chapter we present a conditional theorem which gives the generating series of the differences of contributions of n-cylinder square-tiled surfaces to the spin connected components of the minimal strata $\mathcal{H}(2g-2)$ of Abelian differentials. This series is a refinement of the generating series for the total volume differences obtain previously by Chen, Möller, Sauvaget and Zagier [CMSZ20] using intersection theory.

The theorem is conditional because it relies on a yet unproven property of certain counting functions. More precisely, we introduce an invariant of face-bipartite ribbon graphs with one vertex called combinatorial spin parity. It is a combinatorial analog of the topological invariant used to distinguish the connected components of the minimal stratum. The unproven property is that the counting functions for the families of ribbon graphs with even/odd combinatorial spin parity have polynomial top-degree terms which are equal (except in one base case).

4.1 Preliminaries

4.1.1 Connected components of minimal strata

Connected components of strata of Abelian differentials have been classified by Kontsevich and Zorich in [KZ03]. In general, each stratum has at most 3 connected components. In this chapter we will only be interested in the case of minimal strata $\mathcal{H}(2g-2)$ for which we have the following classification (the terminology is explained below):

- for $g \geq 4$, $\mathcal{H}(2g-2)$ has 3 connected components: the hyperelliptic one

$\mathcal{H}^{hyp}(2g-2)$, and two other components $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$ corresponding to even and odd spin structures;

- for $g = 3$, $\mathcal{H}(2g-2) = \mathcal{H}(4)$ has two connected components $\mathcal{H}^{hyp}(4)$ and $\mathcal{H}^{odd}(4)$;

- for $g \in \{1, 2\}$, $\mathcal{H}(2g-2)$ is connected and coincides with its hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$.

We now explain how to distinguish different connected components (see [KZ03] for more details).

Hyperellipticity

The *hyperelliptic component* $\mathcal{H}^{hyp}(2g-2)$ consists of Abelian differentials ω with a single zero of multiplicity $2g-2$ on *hyperelliptic* Riemann surfaces X of genus g.

Recall that a Riemann surface X is *hyperelliptic* if it admits a ramified double cover $X \to \mathbb{C}P^1$. Such cover has $2g+2$ ramification points which are necessarily simple. X then admits a *hyperelliptic involution* $\sigma : X \to X$, $\sigma^2 = \text{Id}$, which interchanges the sheets of the cover. It has $2g+2$ fixed points which are the ramification points above.

To get a geometric interpretation of hyperellipticity for the corresponding translation surfaces, note that $\sigma^*\omega = -\omega$. Indeed, this is true for all Abelian differentials on X: if for some ω we have $\sigma^*\omega = \omega$, then ω has a zero at each of the $2g+2$ fixed points (σ acts as $z \mapsto -z$ in an appropriate coordinate z around such point); but ω cannot have more then $2g-2$ zeros, a contradiction.

Now, the pair $(X, -\omega)$ corresponds to the same translation surface as (X, ω) except the vertical direction is changed to the opposite one. Hence, σ is an isometry of (X, ω) as a flat surface. Clearly, the conical singularity (i.e. the zero of ω) is a fixed point of σ. Near this singularity σ acts as a "rotation by $(2g-1)\pi$". Near each of the other $2g+1$ fixed points σ acts as a central symmetry. In particular, (X, ω) can be represented as a centrally-symmetric $4g$-gon with opposite sides identified (the center of the polygon can be any one of the $2g+1$ fixed points of σ distinct from the singularity), see Figure 4.1.

Parity of the spin structure

To distinguish other connected components, we need an invariant called the *parity of the spin structure*. We first give an algebraic definition.

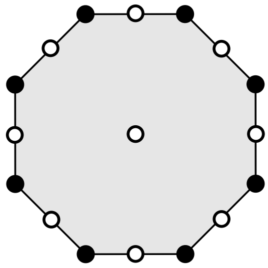


Figure 4.1: A surface of genus $g = 2$ from the hyperelliptic connected component $\mathcal{H}^{hyp}(2)$. The black points represent the singularity, while the white points represent $2g + 1 = 5$ other fixed points of the "central symmetry" σ.

Recall that the *canonical class* $K_X \in \mathrm{Pic}(X)$ of a compact Riemann surface X is the class of divisors corresponding to the sections of the cotangent bundle T^*X. A *spin structure* on a compact Riemann surface X is a choice of a half of the canonical class, i.e. an element $\alpha \in \mathrm{Pic}(X)$ such that $2\alpha = K_X$. Equivalently, it is a choice of a "square root of the cotangent bundle", i.e. a line bundle L such that $L \otimes L \simeq T^*X$. The *parity of the spin structure* α is the residue modulo 2 of the dimension $\dim H^0(X, L)$ of the space of sections of a line bundle L corresponding to α.

Suppose $\omega \in \mathcal{H}(2g-2)$ is an Abelian differential on a surface X, with the zero of multiplicity $2g - 2$ at a point P. Then the divisor $(2g - 2)P$ represents the canonical class, and we have a canonical spin structure on X defined by the divisor $(g - 1)P$. We call the parity of this spin structure the *spin parity of* ω.

It follows from the work of Atiyah [Ati71] and Mumford [Mum71] that the parity of the spin structure is invariant under continuous deformations. Hence, on each connected component of $\mathcal{H}(2g-2)$ the spin parity is constant. The *spin connected components* $\mathcal{H}^{even}(2g - 2)$ and $\mathcal{H}^{odd}(2g - 2)$ consist of differentials on non-hyperelliptic surfaces with even and odd canonical spin parities respectively.

Remark 4.1 ([KZ03], Appendix B)**.** *The spin parity of the hyperelliptic component* $\mathcal{H}^{hyp}(2g - 2)$ *is equal to*

$$\left[\frac{g+1}{2}\right] \pmod 2.$$

Alternative definitions of spin structures

We now give a topological definition of a spin structure.

A spin structure on a smooth surface S is a *double covering of S^1-bundles* $Q \to T_1S$ whose restriction to each fiber is isomorphic to the standard double covering $S^1 \to S^1$ (T_1S is the unit tangent bundle of S for some/any choice of metric). Equivalently, it is a choice of an element $\xi \in H^1(T_1S, \mathbb{Z}/2\mathbb{Z})$ with non-zero value on the class of the fiber.

In the paper [Joh80] Johnson gives one more equivalent definition: a spin structure is a $\mathbb{Z}/2\mathbb{Z}$-valued *quadratic form* q on $H_1(S, \mathbb{Z}/2\mathbb{Z})$ endowed with its intersection form. This means that

$$q(a+b) = q(a) + q(b) + a \cdot b, \;\; a, b \in H_1(S, \mathbb{Z}/2\mathbb{Z}), \tag{4.1}$$

where the dot denotes the intersection form.

A quadratic form q_ξ corresponding to $\xi \in H^1(T_1S, \mathbb{Z}/2\mathbb{Z})$ is defined as follows. For a class $a \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ choose a representative $\alpha = \sum_{i=1}^n \alpha_i$, where α_i are simple closed oriented curves on S. Let $\vec{\alpha_i}$ be the curve in T_1S consisting of the tangent unit vectors to α_i. Johnson proves that the class $\widetilde{a} = \sum_{i=1}^n [\vec{\alpha_i}] \in H_1(T_1S, \mathbb{Z}/2\mathbb{Z})$ does not depend on the choice of the representative α of a. The quadratic form is then given by

$$q_\xi(a) = \langle \xi, \widetilde{a} \rangle, \tag{4.2}$$

where angle brackets denote the duality pairing.

Johnson also proves that the parity of the spin structure ξ coincides with the *Arf-invariant* of q_ξ, which is defined as

$$\mathrm{Arf}(q_\xi) = \sum_{i=1}^{g} q_\xi(a_i) q_\xi(b_i), \tag{4.3}$$

for any choice of a symplectic basis a_i, b_i of $H_1(S, \mathbb{Z}/2\mathbb{Z})$.

Computing spin parity defined by a differential

Consider a surface X with an Abelian differential ω. Recall that the flat structure defined by ω has a consistent choice of the direction to the north at each point of X except the zeros of ω. Let now α be a simple closed oriented curve on X that omits the zeros of ω. Then we can define its *index* $\mathrm{ind}_\omega(\alpha) \in \mathbb{Z}$ as the total change of the angle between its tangent vector and the direction to the north divided by 2π. Index is extended to linear combinations of curves by linearity.

As noted in [KZ03], the spin structure ξ_ω defined by an Abelian differential ω has the following property:

$$\langle \xi_\omega, [\vec{\alpha}] \rangle = \mathrm{ind}_\omega(\alpha) + 1 \pmod 2.$$

Taking into account (4.2) and (4.3), we thus get

Proposition 4.2. *Let (X, ω) be a translation surface. Then the spin parity of (X, ω) is equal to the Arf-invariant of the quadratic form q_ω on $H_1(X, \mathbb{Z}/2\mathbb{Z})$ satisfying $q_\omega([\alpha]) = \mathrm{ind}_\omega(\alpha) + 1 \pmod 2$ for any simple closed curve α on X (Arf invariant is defined in (4.3)).*

4.1.2 Volumes of connected components

Hyperelliptic components

The following explicit expression for the volumes of the hyperelliptic components follows from the work of Athreya, Eskin and Zorich in [AEZ16]. More precisely, Theorem 1.1 of this paper gives explicit expressions for the volumes of all strata of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}P^1$. Combining it with the isomorphism

$$\mathcal{Q}_g(2g-3, (-1)^{2g+1}) \cong \mathcal{H}^{hyp}(2g-2)$$

given by the canonical double cover construction, and taking into account the labeling of the poles (see Remark 1.2 of the same paper), we get:

Theorem 4.3 ([AEZ16])**.** *For all $g \geq 1$:*

$$\mathrm{Vol}(\mathcal{H}^{hyp}(2g-2)) = \frac{2}{(2g+1)!}\frac{(2g-3)!!}{(2g-2)!!}\pi^{2g}.$$

Spin components

The volumes of spin connected components of all strata (admitting such components) were computed in the work of Chen, Möller, Sauvaget and Zagier [CMSZ20]. Following these authors we define the *even (odd) spin subspace* of $\mathcal{H}(2g-2)$ as the union of connected components of $\mathcal{H}(2g-2)$ with even (odd) spin parity. As manifested by Remark 4.1 above, the hyperelliptic component belongs to one subspace or the other depending on g. We define the volume of the subspace as the sum of the volumes of constituent connected components.

The following formula is given[1] in Proposition 11.1 in [CMSZ20].

Theorem 4.4 ([CMSZ20])**.** *Let $\mathrm{Vol}^{\Delta}(\mathcal{H}(2g-2))$ denote the difference between the volumes of even and odd subspaces of $\mathcal{H}(2g-2)$. Then*

$$\mathrm{Vol}^{\Delta}(\mathcal{H}(2g-2)) = \frac{2(2\pi i)^{2g}}{(2g-1)!}d_g,$$

[1] Note that the definition of the series P_Z in the formula (114) of [CMSZ20] has a typo, the correct definition is given in Section 6.3; on the other hand, there is a missing factor of 2 in the Corollary 6.11 of that section).

where $d_g = \frac{1}{2g-1} \cdot [t^{2g}] \frac{t}{(t/P_Z(t))^{-1}}$ *and*

$$P_Z(t) = \exp\left(\sum_{i \geq 1} \left(\frac{1}{2}\right)^i \zeta(-2i+1) t^{2i} \right). \tag{4.4}$$

Since the volumes of $\mathcal{H}(2g-2)$ and $\mathcal{H}^{hyp}(2g-2)$ are known, as well as the spin parity of $\mathcal{H}^{hyp}(2g-2)$ as a function of g (Remark 4.1), Theorem 4.4 indeed allows to compute the volumes of $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$.

4.2 Summary of results

Cylinder contributions for square-tiled surfaces

Conditional theorem 4.5. *The difference* $\mathrm{Vol}_n^{\Delta}(2g-2)$ *of the contributions of* n*-cylinder square-tiled surfaces to the volumes of even and odd spin subspaces of the minimal stratum* $\mathcal{H}(2g-2)$ *is equal to* $\frac{2(2\pi i)^{2g}}{(2g-1)!} d_{g,n}$*, where the numbers* $d_{g,n} \in \mathbb{Q}$*, and whose bivariate generating function*

$$\mathcal{D}(t,u) = 1 + \sum_{g \geq 1} \left(\sum_{n=1}^{g} d_{g,n} u^n \right) (2g-1) t^{2g}$$

satisfies for all $k \geq 1$

$$\frac{1}{2k} [t^{2k}] \mathcal{D}(t,u)^{2k} = \frac{B_{2k}}{2^{k+1} k} u, \tag{4.5}$$

where B_{2k} *is the* $2k$*-th Bernoulli number.*

Using Lagrange inversion, (4.5) is equivalent to

$$\mathcal{D}(t,u) = \frac{t}{Q^{-1}(t,u)}, \quad Q(t,u) = t \cdot \exp\left(u \sum_{k=1}^{\infty} \frac{B_{2k}}{2^{k+1} k} t^{2k} \right), \tag{4.6}$$

where the functional inversion is with respect to the variable t.

Setting $u = 1$ in (4.6), we recover the formula of Chen, Möller, Sauvaget and Zagier for the total volume differences, given in Theorem 4.4. Indeed, the two series $Q(t,1)$ and $t/P_Z(t)$ are equal since for $k \geq 1$ we have $\zeta(-2k+1) = \frac{-B_{2k}}{2k}$. Hence

$$d_g = \sum_{n=1}^{g} d_{g,n} = \frac{1}{2g-1} \cdot [t^{2g}] \mathcal{D}(t,1) = \frac{1}{2g-1} \cdot [t^{2g}] \frac{t}{(t/P_Z(t))^{-1}}.$$

We present in Tables 4.1 and 4.2 the values of $d_{g,n}$ and $\mathrm{Vol}_n^{\Delta}(2g-2)$ for small g.

$g\backslash n$	1	2	3	4
1	$\frac{1}{24}$			
2	$-\frac{1}{1440}$	$-\frac{1}{1152}$		
3	$\frac{1}{10080}$	$\frac{1}{11520}$	$\frac{5}{82944}$	
4	$-\frac{1}{26880}$	$-\frac{221}{9676800}$	$-\frac{7}{552960}$	$-\frac{49}{7962624}$

Table 4.1: Values of the normalized differences of volume contributions $d_{g,n}$ for $g \leq 4$.

$g\backslash n$	1	2	3	4
1	$-\frac{1}{3}\pi^2$			
2	$-\frac{1}{270}\pi^4$	$-\frac{1}{216}\pi^4$		
3	$-\frac{1}{9450}\pi^6$	$-\frac{1}{10800}\pi^6$	$-\frac{1}{15552}\pi^6$	
4	$-\frac{1}{264600}\pi^8$	$-\frac{221}{95256000}\pi^8$	$-\frac{1}{777600}\pi^8$	$-\frac{7}{11197440}\pi^8$

Table 4.2: Values of the differences of volume contributions $\mathrm{Vol}_n^\Delta(2g-2)$ for $g \leq 4$.

Remark 4.6. *The values in Table 4.2 suggest that the contributions to odd subspaces are always bigger than the contributions to the corresponding even subspaces. We conjecture that* $\mathrm{Vol}_n^\Delta(2g-2) < 0$ *for all* $g, n \geq 1$.

Combinatorial spin parity

Theorem 4.5 above relies on the following unproven claim.

Conjecture 4.7. *There exists a collection of functions* $\phi : \mathcal{E}_{g,n,n} \to \mathbb{Z}/2\mathbb{Z}$ *for all* $g \geq 0, n \geq 1$*, which we call* combinatorial spin parity*, such that:*

- *if* G *comes from the cylinder decomposition of a square-tiled surface* $S \in \mathcal{H}(2g-2)$ *for some* g *(see proof of Proposition 3.13), then* $\phi(G)$ *coincides with the spin parity of* S*;*
- *let* $\mathcal{P}_{n,n}^{g,0}$ *and* $\mathcal{P}_{n,n}^{g,1}$ *denote the counting functions of the families* $\{G \in \mathcal{E}_{g,n,n} : \phi(G) = 0\}$ *and* $\{G \in \mathcal{E}_{g,n,n} : \phi(G) = 1\}$ *respectively. Then:*

– *if $n \neq 1$, then for $\varepsilon \in \{0,1\}$ and $(L,L') \in H^\circ_{n,n} \cap (\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0})$:*

$$top\left(\mathcal{P}^{g,\varepsilon}_{n,n}\right)(L,L') = \frac{1}{2} P^g_{H_{n,n}}(L,L'). \tag{4.7}$$

– *For $n = 1$, $\varepsilon = 0, 1$ and $(L,L') \in H^\circ_{n,n} \cap (\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0})$ we have:*

$$top\left(\mathcal{P}^{g,\varepsilon}_{1,1}\right)(L,L') = \frac{2^g + (-1)^{\varepsilon+1} \cdot (g+1)}{2^{g+1}} P^g_{H_{1,1}}(L,L'). \tag{4.8}$$

- *if $G \in \mathcal{E}_{g,n,n}$ is such that the dual G^* consists of two graphs G_1^* and G_2^* connected by a bridge, and such that the sets of black and white labels of vertices are equal in G_1^* and in G_2^*, then*

$$\phi(G) = \phi(G_1) + \phi(G_2). \tag{4.9}$$

Here $P^g_{H_{k,l}}$ is the top-degree term of the counting function $\mathcal{P}^g_{k,l}$ on $H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$. Its expression is given in Proposition 3.3.

Remark 4.8. *In fact such maps ϕ as in Conjecture 4.7 should exist for all $\mathcal{E}_{g,k,l}$ with k and l of equal parity. In this case the corresponding two counting functions' top-degree terms should also be given by $\frac{1}{2} P^g_{H_{k,l}}(L,L')$.*

More details on Conjecture 4.7 and some partial results are presented in section 4.3.

Top degree-term on V_n

Assuming Conjecture 4.7, let us now introduce the difference

$$\mathcal{P}^{g,\Delta}_{n,n}(L,L') = \mathcal{P}^{g,0}_{n,n}(L,L') - \mathcal{P}^{g,1}_{n,n}(L,L').$$

Recall that the subspace V_n of $\mathbb{R}^n \times \mathbb{R}^n$ is defined by the set of equations $\{L_1 = L_1', \ldots, L_n = L_n'\}$. The top-degree term of $\mathcal{P}^{g,\Delta}_{n,n}$ on V_n° also turns out to be polynomial, and we have the following explicit formula for it, which is an analog of Theorem 3.4. This formula will imply the Conditional theorem 4.5.

Conditional theorem 4.9. *For all $g \geq 0$, $n \geq 1$, the top-degree term of $\mathcal{P}^{g,\Delta}_{n,n}(L,L')$ on $H^+_{n,n} \cap V_n^\circ$ is polynomial. It is given by the following symmetric polynomial of degree $2g$:*

$$P^{g,\Delta}_{V_n}(L,L) = \sum_{\substack{s_1+\ldots+s_n = g+n \\ s_i \geq 1}} p^\Delta_{s_1,\ldots,s_n} L_1^{2s_1-2} \cdots L_n^{2s_n-2},$$

where the numbers $p^{\Delta}_{s_1,\ldots,s_n} \in \mathbb{Q}$ *and whose generating function*

$$\mathcal{T}^{\Delta}(t,t_1,t_2,\ldots) = 1 + \sum_{s,n\geq 1}(2s-1)t^{2s}\frac{1}{n!}\sum_{\substack{s_1+\ldots+s_n=s\\ s_i\geq 1}} p^{\Delta}_{s_1,\ldots,s_n}t_{s_1}\cdots t_{s_n},$$

satisfies the following relation for all $k \geq 1$*:*

$$\frac{1}{2k}[t^{2k}]\mathcal{T}^{\Delta}(t,t_1,t_2,\ldots)^{2k} = -\frac{t_k}{2^{k-1}}. \tag{4.10}$$

Using a version of Lagrange inversion (see formula (2.2.9) in [Ges16]), (4.10) can be rewritten as

$$\mathcal{T}^{\Delta}(t,t_1,\ldots) = \frac{t}{Q^{-1}(t,t_1,\ldots)},\ Q(t,t_1,\ldots) = t\cdot\exp\left(-\sum_{k=1}^{\infty}\frac{t_k}{2^{k-1}}t^{2k}\right),$$

where the functional inversion is with respect to the variable t.

We present in Table 4.3 the values of $p^{\Delta}_{s_1,\ldots,s_n}$ for small values of s_i.

p_1	p_2	$p_{1,1}$	p_3	$p_{2,1}$	$p_{1,1,1}$	p_4	$p_{3,1}$	$p_{2,2}$	$p_{2,1,1}$	$p_{1,1,1,1}$
-1	$-\frac{1}{6}$	-1	$-\frac{1}{20}$	$-\frac{1}{2}$	-5	$-\frac{1}{56}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{7}{2}$	-49

Table 4.3: Values of $p^{\Delta}_{s_1,\ldots,s_n}$ with $s_1+\ldots+s_n \leq 4$.

4.3 Combinatorial spin parity

Case $n = 1$

First of all, the $n = 1$ case of Conjecture 4.7 is not difficult to prove. Indeed, every graph $G \in \mathcal{E}_{g,1,1}$ comes from a cylinder decomposition of some square-tiled surfaces from $\mathcal{H}(2(g+1)-2)$ (the two boundary components of any such graph automatically have equal perimeters for any metric, so it is enough to glue one cylinder to obtain a square-tiled surface). So we can define ϕ as the spin parity of any corresponding square-tiled surface (it does not depend on the size or the twist of the cylinder).

Moreover, the contribution of each graph to the top-degree term of the counting function is easily seen to be equal to $L_1^{2g}/(2g)!$. It means that the proof 4.7 boils down to counting ribbon graphs $\mathcal{E}_{g,1,1}$ with given spin parity. This is can be done using a simple recursion from [Del13, section 4.3.3].

Case $g = 0$

We understand fairly well the case $g = 0$. Recall that the dual graphs of graphs in $\mathcal{E}_{0,n,n}$ are simply vertex-bicolored plane trees with labeled vertices. We work here with the duals (recall the duality Convention 1.7).

Suppose a tree $T \in \mathcal{E}^*_{0,g,g}$ is the dual of a ribbon graph coming from some square-tiled surface X of genus g. We have the following combinatorial interpretation of the spin parity of X in terms of T.

Lemma 4.10. *For each $i = 1, \dots g$ choose an oriented path γ_i between the white and the black vertices of T with labels i along the boundary of the unique face. Then the parity of the spin structure of X is equal to*

$$\sum_{i=1}^{g} \frac{l(\gamma_i) + 1}{2} + |S| \pmod 2, \tag{4.11}$$

where $l(\gamma)$ is the (graph-theoretic) length of the path γ, and $S = \{(i, j) : i < j,\ \text{exactly 1 endpoint of } \gamma_i \text{ lies inside } \gamma_j\}$.

Proof. First, we choose a (non-symplectic) basis $(\alpha_i, \beta_i)_{i=1,\dots,g}$ of $H_1(X, \mathbb{Z})$ as follows. α_i is a horizontal waist curve of the i-th cylinder. Let $0 < \varepsilon_1 < \dots < \varepsilon_g$ be sufficiently small numbers. For each i, let β_i

- go around the singularity at a constant distance ε_i from it, through the corners corresponding to the inner vertices of the path γ_i;
- when it reaches the bottom (black) boundary of the i-th cylinder, it goes vertically upwards inside the i-th cylinder and comes back to the point on the top (white) boundary where it started (we can assume this since twisting the cylinders does not change the parity).

See Figure 4.2 for an illustration.

Let $\langle \cdot, \cdot \rangle$ be the intersection form on $H_1(X)$. Obviously, for all i, j we have $\langle \alpha_i, \alpha_j \rangle = 0$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. When $i < j$, $\varepsilon_i < \varepsilon_j$, so the curves β_i and β_j can intersect (transversely) only at a point where β_i goes vertically inside i-th cylinder and β_j goes circularly around the singularity. Each such point corresponds to the endpoint of γ_i lying inside γ_j.

Consider the basis $(\alpha_i, \beta'_i)_{i=1,\dots,g}$ of $H_1(X)$, where

$$\begin{aligned}
\beta'_1 &= \beta_1 - \langle \beta_1, \beta_2 \rangle \, \alpha_2 - \langle \beta_1, \beta_3 \rangle \, \alpha_3 - \dots - \langle \beta_1, \beta_g \rangle \, \alpha_g, \\
\beta'_2 &= \beta_2 - \langle \beta_2, \beta_3 \rangle \, \alpha_3 - \dots - \langle \beta_2, \beta_g \rangle \, \alpha_g, \\
&\dots \\
\beta'_{g-1} &= \beta_{g-1} - \langle \beta_{g-1}, \beta_g \rangle \, \alpha_g, \\
\beta'_g &= \beta_g.
\end{aligned}$$

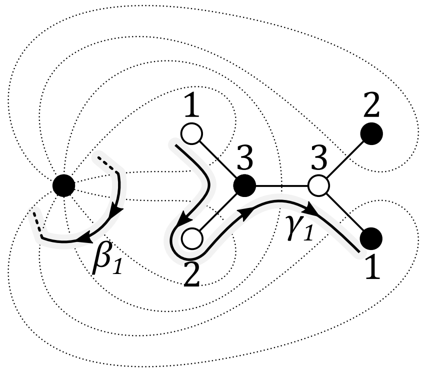


Figure 4.2: Construction of the curve β_i.

Then, for $i < j$ we have $\langle \beta'_i, \beta'_j \rangle = \langle \beta_i, \beta_j \rangle - \langle \beta_i, \beta_j \rangle \langle \alpha_j, \beta_j \rangle = 0$. So this new basis is symplectic. We apply Proposition 4.2 to this basis. Note that, since α_i are horizontal curves, $\mathrm{ind}(\alpha_i) = 0$ and so $q([\alpha_i]) = 1$. The curve β_i is vertical inside i-th cylinder and turns by π inside each corner around the singularity. The number of such corners is $l(\gamma_i) - 1$, so $\mathrm{ind}(\beta_i) = \frac{l(\gamma_i)-1}{2}$. Using (4.1) we see that $q([\beta'_i]) = \frac{l(\gamma_i)-1}{2} + 1 + |\{j : j > i, \langle \beta_i, \beta_j \rangle \text{ is odd}\}|$, since we look at α_i and β'_i as elements of $H_1(X, \mathbb{Z}/2\mathbb{Z})$. Substituting, we get (4.11). □

Now, there is a degree of freedom in choosing the γ_i in Lemma 4.10. We can use this to give a simpler expression for the spin parity. For this we can use the prefix-postfix marking and prefix-postfix sequence of trees, which are introduced in section 5.1 of Chapter 5.

Lemma 4.11. *The spin parity of X coincides with the parity of the permutation given by the prefix-postfix sequence $\pi(T)$, where T is rooted at an arbitrary black corner and the parity of the permutation is computed with respect to the base sequence $(1^\bullet 1^\circ 2^\bullet 2^\circ \dots g^\bullet g^\circ)$.*

Proof. Root T at an arbitrary black corner. Consider the prefix-postfix marking of T. For each $i = 1, \dots, g$, choose γ_i in Lemma 4.10 to be the path going from the first-visit marker of the black vertex i to the last-visit marker of the white vertex i, and not passing through the root corner.

Lemma 5.11 implies that the distance $l(\gamma_i)$ is equal to:

- 1 plus two times the number of markers between the endpoint markers of γ_i, if black endpoint of γ_i comes before the white one (counterclockwise around T);
- 3 plus two times the number of markers between the endpoint markers of γ_i, otherwise.

Using this, we can easily see that (4.11) is equal to to the number of elementary transpositions needed to transform $\pi(T)$ into the base sequence $(1^\bullet 1^\circ 2^\bullet 2^\circ \ldots g^\bullet g^\circ)$. It is well known that the parity of this number is equal to the parity of the permutation $\pi(T)$. □

Using this description, we can define $\phi(G)$ for any $G \in \mathcal{E}_{0,k,l}$ with k, l of equal parity, as the parity of the prefix-postfix sequence of G^* with an arbitrary rooting. One can show that the parity condition on k, l implies that this does not depend on the choice of the rooting (we omit the proof).

With this definition, the formula (4.7) of Conjecture 4.7 becomes easy to proof. Theorem 5.3 states that at any point $(L, L') \in H^\circ_{n,n} \cap (\mathbb{R}^n_{>0} \times \mathbb{R}^n_{>0})$ there is exactly one positive tree with given (cyclic equivalence class of) prefix-postfix sequence. One can thus obtain the bijection between even and odd spin parity trees positive at a fixed point (L, L') by simply switching two fixed black (or white) labels in the prefix-postfix sequence.

Case $g > 0$

The case of positive genus ribbon graphs is less understood.

It is still possible to give a combinatorial interpretation of the spin parity of the surface in terms of the corresponding ribbon graph, similar to the one in Lemma 4.10. One can compute the corresponding expression for any graph $G \in \mathcal{E}_{g,n,n}$, even if it does not come from the cylinder decomposition of a square-tiled surface. By defining $\phi(G)$ to be equal to this expression we were able to check Conjecture 4.7 for small values of g, n ($g \leq 2$, $n \leq 3$).

This obtained expression for the spin parity is quadratic in the combinatorial data of the graph. The reason is that the ribbon graph itself has non-trivial topology now, and so the curves α_i, β_i should be complemented with curves forming a symplectic basis of the ribbon graphs itself.

To prove (4.7) with this definition of ϕ, one has to be able to control both the spin parity of a ribbon graph and the vertex perimeters of its metric.

One approach would be to emulate the proofs of Chapter 3, in particular to use the Chapuy-Féray-Fusy bijection. However, because of its implicit nature, one cannot control the spin parity.

Another approach would be to try to use the genus 0 case technique described above. For example, one can cut vertices of the dual graph G^* to obtain a tree, modify this tree using the exchange of labels in the prefix-postfix sequence (which preserves the vertex perimeters), and reglue the tree back. We were able to construct such transformations which change the spin parity and preserve the vertex perimeters of the metric. However, we have not been able (yet) to find such construction which is canonical (to be able to use it for enumeration purposes).

4.4 Proof of Conditional theorem 4.9

The proof of Theorem 4.9 is similar to that of Theorem 3.4. The main difference is the following. In the proof of Theorem 3.4 (section 3.3.2) we: (a) proved a recurrence relation for the numbers $p_{s_1,\dots,s_n}$ using their combinatorial interpretation as cardinalities of certain families of metric plane trees (Proposition 3.11); (b) using the Chapuy-Féray-Fusy bijection, proved that the top-degree term of $\mathcal{P}^g_{n,n}$ on $H^+_{n,n} \cap V^\circ_n$ is a polynomial (denoted by $P^g_{V_n}$) and that $p_{s_1,\dots,s_n}$ are in fact the coefficients of $P^g_{V_n}$ (Theorem 3.2).

For the proof of Theorem 4.9 we would like to avoid using the aforementioned bijection, since it makes it difficult to control the combinatorial spin parity of the graph. To this end, we directly prove a recurrence relation for the top-degree terms of $\mathcal{P}^{g,\Delta}_{n,n}$ on $H^+_{n,n} \cap V^\circ_n$ (Proposition 4.13) which will simultaneously: (a) imply that these top-degree terms are symmetric polynomials (which we will denote by $P^{g,\Delta}_{V_n}$); (b) by coefficient extraction, provide the recurrence relation for the coefficients of $P^{g,\Delta}_{V_n}$ (which we will denote by $p^\Delta_{s_1,\dots,s_n}$).

Remark 4.12. *By analogy, a question arises whether the (properly normalized) numbers $p^\Delta_{s_1,\dots,s_n}$ are differences of cardinalities of families of even/odd-spin metric plane trees. One can show (we omit the details) that those differences do satisfy the same recurrence relation (4.19) as the numbers $p^\Delta_{s_1,\dots,s_n}$, but their initial conditions differ (see (4.18) for the initial condition for the numbers $p^\Delta_{s_1,\dots,s_n}$).*

Proposition 4.13. *The top-degree term of $\mathcal{P}^{g,\Delta}_{1,1}$ on $H^+_{1,1} \cap V^\circ_1$ is given by*

$$top(\mathcal{P}^{g,\Delta}_{1,1})(L;L) = -\frac{g+1}{2^g} P^g_{H_{1,1}}(L,L) = -\frac{L^{2g}}{2^g(2g+1)}. \tag{4.12}$$

For $n \geq 2$ the top-degree terms of $\mathcal{P}^{g,\Delta}_{n,n}$ on $H^+_{n,n} \cap V^\circ_n$ satisfy the following recurrence relation for any $(L;L) = (L_1,\dots,L_n;L_1,\dots,L_n)$ in $H^+_{n,n} \cap V^\circ_n$:

$$0 = top\left(\mathcal{P}^{g,\Delta}_{n,n}\right)(L;L) + \sum_{t=2}^{n} \frac{(2g+2n-2)_{t-2}}{t!} \times$$
$$\times \sum_{I_1,\dots I_t} \sum_{\substack{g_1+\dots+g_t=g \\ g_i \geq 0}} \prod_{j=1}^{t} (2g_j + 2|I_j| - 1) \cdot top\left(\mathcal{P}^{g_j,\Delta}_{|I_j|,|I_j|}\right)(L_{I_j};L_{I_j}), \tag{4.13}$$

where as before $(x)_t := x(x-1)\cdots(x-t+1)$ is the falling factorial; the second sum is over all partitions of $\{1,\dots,n\}$ into t non-empty labeled sets $I_1,\dots,I_t$; and L_{I_j} denotes $(L_i)_{i\in I_j}$.

Proof of Proposition 4.13. First note that for $n=1$ we automatically have $L_1 = L_1'$, i.e. $H_{1,1} = V_1$, and hence

$$top(\mathcal{P}^{g,\Delta}_{1,1})(L;L) = -\frac{g+1}{2^g} P^g_{H_{1,1}}(L,L) = -\frac{L^{2g}}{2^g(2g+1)}.$$

by (4.8) and Proposition 3.3.

Let now $n\geq 2$. Equation (4.13) will follow from substituting $\varepsilon=0$ and $\varepsilon=1$ in the following equation, and then taking the difference:

$$\frac{1}{2}P^g_{H_{n,n}}(L,L) = top\left(\mathcal{P}^{g,\varepsilon}_{n,n}\right)(L;L) + \sum_{t=2}^{n}\frac{(2g+2n-2)_{t-2}}{t!}\times$$
$$\times \sum_{I_1,\dots I_t}\ \sum_{\substack{g_1+\ldots+g_t=g\\ \varepsilon_1+\ldots+\varepsilon_t=\varepsilon \pmod 2\\ g_i\geq 0,\ \varepsilon_i\in\{0,1\}}} \prod_{j=1}^{t}(2g_j+2|I_j|-1)\cdot top\left(\mathcal{P}^{g_j,\varepsilon_j}_{|I_j|,|I_j|}\right)(L_{I_j};L_{I_j}). \quad (4.14)$$

On the right-hand side, we use the definition $\mathcal{P}^{g_j,\Delta}_{|I_j|,|I_j|} = \mathcal{P}^{g_j,0}_{|I_j|,|I_j|} - \mathcal{P}^{g_j,1}_{|I_j|,|I_j|}$.

To prove (4.14), consider the degeneration

$$(L_1+(n-1)\epsilon, L_2-\epsilon,\dots,L_n-\epsilon; L_1,\dots,L_n),\ \epsilon\to 0, \epsilon\geq 0 \qquad (4.15)$$

from the ambient space $H_{n,n}$ to the subspace V_n.

By Lemma 2.22 we have

$$top\left(\mathcal{P}^{g,\varepsilon}_{n,n}\right)(L,L') = \sum_{\substack{G\in\mathcal{E}^*_{g,n,n}\\ \varphi(G)=\varepsilon}} \frac{1}{|\operatorname{Aut}(G)|}\cdot\left(\prod_{e\in B(G)}\mathbf{1}_{f_e(L,L')>0}\right)\cdot \operatorname{Vol} M_G(L,L'), \quad (4.16)$$

where $B(G)$ denotes the set of bridges of G, f_e is the linear form giving the length of the bridge e, and $M_G(L,L')$ is the corresponding polytope of metrics inside the space of weight functions on G.

While the volume functions $\operatorname{Vol} M_G(L,L')$ are continuous in L,L', the indicator functions are not. In particular, the function $top\left(\mathcal{P}^{g,\varepsilon}_{n,n}\right)$ will have a discontinuity at $\epsilon=0$ along the path (4.15).

The relation (4.14) precisely quantifies the jump of $top\left(\mathcal{P}^{g,\varepsilon}_{n,n}\right)$ at $\epsilon=0$. Indeed, on the left hand side we have the polynomial giving the limit value

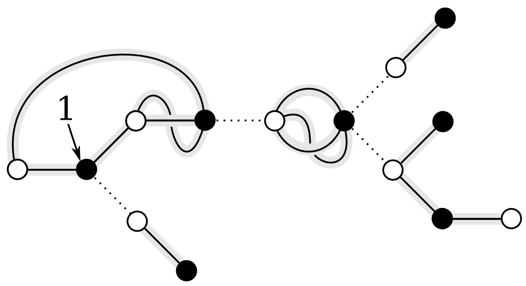


Figure 4.3: A degenerate graph with $t = 5$ components. The zero-length bridges are dotted. We have marked the black vertex number 1.

of $top\left(\mathcal{P}_{n,n}^{g,\varepsilon}\right)$ when $\epsilon \to 0$, the actual value is of course $top\left(\mathcal{P}_{n,n}^{g,\varepsilon}\right)(L;L)$, and the remaining sum is the jump.

The jump is the sum of contributions of graphs G which are positive for $\epsilon > 0$ (i.e. their contribution is not identically zero) and which have at least one bridge whose length becomes zero at $\epsilon = 0$. We call such graphs degenerate. We claim that such graphs G are characterized by being constructed as follows:

1. choose a partition $I_1, \ldots, I_t$ of $\{1, \ldots, n\}$ into $t \geq 2$ non-empty sets;
2. choose a non-negative composition $g_1 + \ldots + g_t = g$ and a binary vector $(\varepsilon_1, \ldots, \varepsilon_t)$ with $\varepsilon_1 + \ldots + \varepsilon_t = \varepsilon \pmod 2$;
3. for each $j = 1, \ldots, t$ choose a graph $G_j \in \mathcal{E}^*_{g_j,|I_j|,|I_j|}$ with black and white vertex labels in I_j, which is positive at the point $(L_{I_j}; L_{I_j})$ and such that $\varphi(G_j) = \varepsilon_j$;
4. construct the graph G by connecting the graphs G_j by bridges into a tree like structure (see Figure 4.3), respecting the following two conditions:
   (a) the bridges that are incident to the graph G_j containing the black vertex number 1 are incident only to its black vertices;
   (b) for all the graphs G_j that do not contain the black vertex number 1, there is exactly one incident bridge which is incident to its white vertex, and several (maybe none) incident bridges which are incident to its black vertices.

Points 1, 3 and 4 follow by the same reasoning as in the proof of Proposition 3.11. The additional constraints we need to include in this context have to do with the genera and the combinatorial spin parities of the pieces G_j.

These constraints are given in point 2. The condition $g_1 + \ldots + g_t = g$ is clear, while the condition $\varepsilon_1 + \ldots + \varepsilon_t = \varepsilon \pmod 2$ is equivalent to $\varphi(G_1) + \ldots + \varphi(G_t) = \varphi(G) \pmod 2$ and is necessary by (4.9).

Let us now study the contribution of any degenerate graph G to the jump. First of all, for such graphs $|\operatorname{Aut}(G)| = 1$: any automorphism f preserves vertex labels, so f must preserve (setwise) the pieces G_j; since the bridges joining the G_j form a tree-like structure, they must be fixed by f; hence each G_j has a corner fixed by f, and so f fixes each G_j, i.e. f is the identity.

Now, the limit value of the contribution of G is $\operatorname{Vol} M_G(L; L)$ (we just disregard the $1/|\operatorname{Aut}(G)|$ and the indicator functions in (4.16)). Clearly, the polytope $M_G(L; L)$ has the product structure $\prod_{j=1}^{t} M_{G_j}(L_{I_j}; L_{I_j})$ and so

$$\operatorname{Vol} M_G(L; L) = \prod_{j=1}^{t} \operatorname{Vol} M_{G_j}(L_{I_j}; L_{I_j}). \tag{4.17}$$

The points 1-3 above, as well as equation (4.17) explain the sums, the products and the terms *top* $\left(\mathcal{P}^{g_j,\varepsilon_j}_{|I_j|,|I_j|}\right)(L_{I_j}; L_{I_j})$ in equation (4.14). The factor $1/t!$ accounts for the arbitrariness of the numbering of the pieces G_j. What is left to explain is the term $(2g + 2n - 2)_{t-2} \cdot \prod_{j=1}^{t}(2g_j + 2|I_j| - 1)$. This term represents the number of ways to join the pieces G_j with the bridges respecting conditions 4a and 4b above. The computation of this term is identical to the computation of the term (3.15) in the proof of Proposition 3.11, except that here the number of corners of the piece G_j adjacent to its black/white vertices is equal to (the number of edges of G_j, which is equal to) $2|I_j| + 2g_j - 1$, so the numbers c_j in that computation should be set to $2|I_j| + 2g_j$. This concludes the proof of (4.14) and hence of the recurrence relation (4.13). □

Proof of Theorem 4.9. Using the recurrence relation of Proposition 4.13 we now prove that the top-degree terms of $\mathcal{P}^{g,\Delta}_{n,n}$ on $H^+_{n,n} \cap V^\circ_n$ are in fact symmetric polynomials of degree $2g$ in the variables L_i^2. We do this by induction on n.

The base case $n = 1$ clearly follows from (4.12). Suppose now that the statement is true for all $k < n$, with $n \geq 2$. Then in equation (4.13) the terms *top* $\left(\mathcal{P}^{g_j,\Delta}_{|I_j|,|I_j|}\right)(L_{I_j}; L_{I_j})$ are symmetric polynomials of degree $2g_j$ in $(L_i^2)_{i \in I_j}$ (because $|I_j| < n$). This implies that *top* $\left(\mathcal{P}^{g,\Delta}_{n,n}\right)$ is a polynomial of degree $2g$ in $(L_i^2)_{1 \leq i \leq n}$. Moreover, it is symmetric, because for each $t = 2, \ldots, n$, the second line of (4.13) is symmetric in L. Indeed, suppose we exchange L_1 and L_2. If 1 and 2 belong to the same I_j, the corresponding term does not change since *top* $\left(\mathcal{P}^{g_j,\Delta}_{|I_j|,|I_j|}\right)(L_{I_j}; L_{I_j})$ is symmetric by induction hypothesis. If 1 and 2 belong to different I_j and $I_{j'}$ respectively, then each of the corresponding terms is exchanged with another term were $I_j, I_{j'}$ are replaced by $I_j \cup \{2\} \setminus \{1\}, I_{j'} \cup \{1\} \setminus \{2\}$ respectively. This concludes the induction step.

Denote by $P^{g,\Delta}_{V_n}$ the polynomial giving the top-degree term of $\mathcal{P}^{g,\Delta}_{n,n}(L,L')$ on $H^+_{n,n} \cap V^\circ_n$. For $s_1,\dots,s_n \geq 1$, let $p^\Delta_{s_1,\dots,s_n}$ be the coefficient of the monomial $L_1^{2s_1-2}\cdots L_n^{2s_n-2}$ in $P^{g,\Delta}_{V_n}$. By extracting the coefficient of this monomial in (4.12) and (4.13), we get the following recurrence relation for the numbers $p^\Delta_{s_1,\dots,s_n}$:

$$-\frac{1}{2^{s-1}(2s-1)} = p^\Delta_s \tag{4.18}$$

$$0 = p^\Delta_{s_1,\dots,s_n} + \sum_{t=2}^{n} \frac{(2s-2)_{t-2}}{t!} \cdot \sum_{I_1,\dots I_t} \prod_{j=1}^{t} \left(2\sum_{i\in I_j} s_i - 1\right) p^\Delta_{s_{I_j}}, \tag{4.19}$$

where s_{I_j} denotes $(s_i)_{i\in I_j}$, and $s = \sum_{i=1}^n s_i$. Note that we have replaced $(2g+2n-2)$ and $(2g_j+2|I_j|-1)$ by $(2s-2)$ and $\left(2\sum_{i\in I_j} s_i - 1\right)$ respectively. The first equality follows from the definition of $p^\Delta_{s_1,\dots,s_n}$ and the fact that the degree of $P^{g,\Delta}_{V_n}$ is equal to $2g$. The second equality follows from the same observation applied to $P^{g_j,\Delta}_{V_{|I_j|}}$.

Now fix $s \geq 1$. Multiply (4.19) by $2s(2s-1)\frac{1}{n!}t_{s_1}\cdots t_{s_n}$ and sum over all $n \geq 2$ and all $s_1,\dots,s_n \geq 1$ such that $s_1+\ldots+s_n = s$. Finally, add (4.18) multiplied by $2s(2s-1)t_s$. We get

$$-\frac{s}{2^{s-2}}t_s = \sum_{n\geq 1}\frac{1}{n!}\sum_{\substack{s_1+\ldots+s_n=s\\ s_i\geq 1}}\sum_{t=1}^{n}\binom{2s}{t}\sum_{I_1,\dots I_t}\prod_{j=1}^{t}\left(2\sum_{i\in I_j}s_i-1\right)p^\Delta_{s_{I_j}}\prod_{i\in I_j}t_{s_i}.$$

Changing the order of summation on the right hand side gives

$$\sum_{t=1}^{2s}\binom{2s}{t}\sum_{n\geq t}\frac{1}{n!}\sum_{\substack{s_1+\ldots+s_n=s\\ s_i\geq 1}}\sum_{I_1,\dots I_t}\prod_{j=1}^{t}\left(2\sum_{i\in I_j}s_i-1\right)p^\Delta_{s_{I_j}}\prod_{i\in I_j}t_{s_i}.$$

Performing the same algebraic transformations as at the end of the proof of Theorem 3.4, we see that this is equal to

$$[t^{2s}]\mathcal{T}^\Delta(t,t_1,t_2,\dots)^{2s},$$

and so $\frac{1}{2s}[t^{2s}]\mathcal{T}^\Delta(t,t_1,t_2,\dots)^{2s} = -\frac{t_s}{2^{s-1}}$ for $s \geq 1$, as desired. □

4.5 Proof of Conditional theorem 4.5

Proof of Theorem 4.5. For $g \geq 1$ and $1 \leq n \leq g$, *define* the number $d_{g,n}$ by the equality $\mathrm{Vol}_n^{\Delta}(2g-2) = \frac{2(2\pi i)^{2g}}{(2g-1)(2g-1)!} d_{g,n}$.

Similarly to non-spin case, we can express the differences of volume contributions to spin subspaces in terms of the polynomials $P_{V_n}^{g-n,\Delta}$:

$$\mathrm{Vol}_n^{\Delta}(2g-2) = 2 \cdot 2g \cdot \frac{1}{n!} \cdot \lim_{N \to \infty} \left(N^{-2g} \sum_{\substack{\sum_{i=1}^n h_i L_i \leq N \\ h_i, L_i \in \mathbb{Z}_{>0}}} L_1 \cdots L_n \cdot P_{V_n}^{g-n,\Delta}(L; L) \right).$$

Using Lemma 1.11, this is equal to

$$\frac{2}{(2g-1)!} \cdot \frac{1}{n!} \cdot \sum_{\substack{s_1 + \ldots + s_n = g \\ s_i \geq 1}} p_{s_1,\ldots,s_n}^{\Delta} \prod_{i=1}^n (2s_i - 1)! \zeta(2s_i).$$

Recall that $\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$, where B_{2k} is the $2k$-th Bernoulli number. Hence,

$$\mathrm{Vol}_n^{\Delta}(2g-2) = \frac{2(2\pi i)^{2g}}{(2g-1)!} \cdot \frac{1}{n!} \cdot \sum_{\substack{s_1 + \ldots + s_n = g \\ s_i \geq 1}} p_{s_1,\ldots,s_n}^{\Delta} \prod_{i=1}^n \frac{-B_{2s_i}}{2(2s_i)},$$

and so

$$d_{g,n} = \frac{1}{n!} \cdot \sum_{\substack{s_1 + \ldots + s_n = g \\ s_i \geq 1}} p_{s_1,\ldots,s_n}^{\Delta} \prod_{i=1}^n \frac{-B_{2s_i}}{2(2s_i)}.$$

This means that $\mathcal{D}(t,u)$ is equal to $\mathcal{T}^{\Delta}(t, t_1, t_2, \ldots)$ evaluated at $t_k = \frac{-B_{2k}}{2(2k)} u$, $k \geq 1$. In particular, the relation (4.10) for the series $\mathcal{T}^{\Delta}$ implies for all $k \geq 1$:

$$\frac{1}{2k} [t^{2k}] \mathcal{D}(t,u)^{2k} = \frac{B_{2k}}{2^{k+1} k} u,$$

as desired. □

Chapter 5

Prefix-postfix sequences of trees

In this chapter we introduce an invariant of plane rooted vertex-bicolored trees with labeled vertices, which we call the *prefix-postfix sequence*. This invariant is interesting from several perspectives.

Firstly, we prove that for any (generic) vertex perimeters (L, L') and any (cyclic equivalence class of) prefix-postfix sequence, there is exactly one rooted metric plane tree with these parameters (Theorem 5.3). This can be seen as a generalization of the case $g = 0, W = H_{k,l}$ in Theorem 3.2. Namely, the counting function for such family of trees is constant with value 1 outside of the walls.

Next, we prove that each family of trees with given (class of) prefix-postfix sequence gives rise to a triangulation of the product of two simplices (Theorem 5.6), which are interesting from the point of view of theory of polytopes. We show that such triangulations admit a recursive construction.

Theorem 5.3 will be used in Chapter 6 to prove the polynomiality of the weighted counting functions for planar many-vertex face-bicolored graphs (Theorem 6.5).

Finally, in Chapter 4 we prove that for trees with equal number of black and white vertices, the parity of a permutation canonically associated to the prefix-postfix sequence coincides with the spin parity of this tree (Lemma 4.11).

5.1 Definitions

Recall from section 1.1.1 that a vertex-bicolored ribbon graph is *rooted* if it has a distinguished black corner, denoted by an oriented *root half-edge*

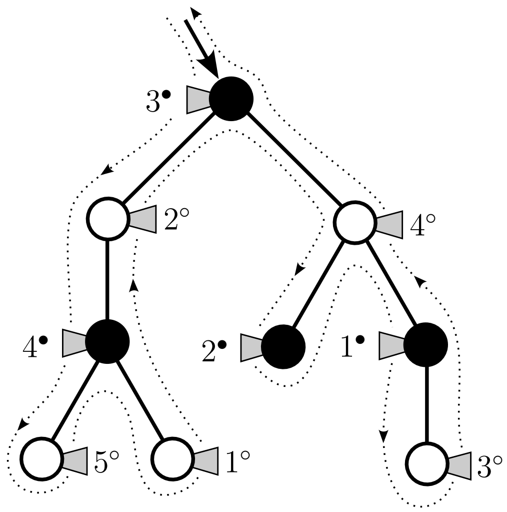


Figure 5.1: Computing the prefix-postfix marking (and sequence) of a tree.

pointing to it. The *root vertex* is the vertex of the root corner and the *root edge* is the edge following the root-half edge clockwise around the root vertex. Recall also that $\mathcal{E}^{*,root}_{0,k,l}$ denotes that set of plane rooted vertex-bicolored trees with k black vertices labeled from 1 to k and l white vertices labeled from 1 to l. In this chapter we will use an alternative notation $\mathcal{T}^{root}_{k,l}$ for this set and, for brevity, we will call its elements simply *trees*. We denote the vertices by their labels with a filled or unfilled circle in the superscript to distinguish black and white vertices respectively. For example, the black vertex number one is denoted by $1^\bullet$ and the white vertex number one is denoted by 1°.

By *going around a tree* T we mean: starting at the root half-edge, following the boundary of T in the counter-clockwise direction, and finishing at the starting point. We say that we *visit a vertex* v of T, if we traverse one of the corners around v.

Definition 5.1 (Prefix-postfix marking)**.** Let $T \in \mathcal{T}^{root}_{k,l}$ be a tree. The *prefix-postfix marking* of T is the marking (distinguishing) of $k + l$ corners of T constructed as follows. Go around T. Every time a black vertex is visited for the first time or a white vertex is visited for the last time, mark the corresponding corner of this vertex. The marking obtained after coming back to the starting point is the prefix-postfix marking of T.

An example of a computation of the prefix-postfix marking is given in Figure 5.1. For visualisation purposes we mark the corners where we visit the corresponding vertex for the first or the last time by a triangular marker. We refer to these corners as *marked corners of* T. For convenience, the *color* *of a marker* and the *color of a corner* are defined to be the color of the corresponding vertex.

Definition 5.2 (Prefix-postfix sequence)**.** Let $T \in \mathcal{T}_{k,l}^{root}$ be a tree. The *prefix-postfix sequence* $\pi(T)$ of T is the sequence constructed as follows. Start with an empty sequence. Go around T. Every time a marked corner is encountered, append the label of the corresponding vertex to the sequence. The sequence obtained after coming back to the starting point is the prefix-postfix sequence of T.

For example, the prefix-postfix sequence of the tree in Figure 5.1 is equal to $(3^\bullet, 4^\bullet, 5^\circ, 1^\circ, 2^\circ, 2^\bullet, 1^\bullet, 3^\circ, 4^\circ)$.

Clearly, $\pi(T)$ is a sequence of length $k+l$ of pairwise distinct elements of the set of labels $\{1^\bullet, \ldots, k^\bullet\} \cup \{1^\circ, \ldots, l^\circ\}$. We avoid referring to it as a permutation of this set, since there is no self-map of this set involved.

Let $\Pi_{k,l}$ be the set of all sequences of lengths $k+l$ whose elements are pairwise distinct and belong to $\{1^\bullet, \ldots, k^\bullet\} \cup \{1^\circ, \ldots, l^\circ\}$. Denote by $\sim$ the equivalence relation ("cyclic equivalence") on $\Pi_{k,l}$ defined by

$$(x_1, \ldots, x_{k+l}) \sim (y_1, \ldots, y_{k+l}) \iff \exists n : y_i = x_{i+n},$$

where indices are modulo $k+l$. The equivalence class of an element $\pi \in \Pi_{k,l}$ is denoted by $[\pi]$.

5.2 Statements of results

5.2.1 Metric trees with given prefix-postfix sequence

Recall the definitions of and the notations for the subspace $H_{k,l} \subset \mathbb{R}^k \times \mathbb{R}^l$, the set of walls $\mathcal{W}_{k,l}$ and the set of their intersections $\overline{\mathcal{W}_{k,l}}$, introduced in section 2.2.1. Recall also that a tree $T \in \mathcal{E}_{0,k,l}^*$ is said to be *positive at the point* $(L, L') \in H_{k,l}$ if the unique (by Lemma 2.5) weight function w on T with $\mathrm{vp}_T(w) = (L, L')$ is positive. We say that a rooted tree $T \in \mathcal{T}_{k,l}^{root}$ *is positive at* (L, L') if the corresponding unrooted tree in $\mathcal{E}_{0,k,l}^*$ is positive at (L, L').

Theorem 5.3. *For every equivalence class $c \in \Pi_{k,l}/\sim$ and for every point $(L; L') \in H_{k,l}^\circ \cap (\mathbb{R}_{>0}^k \times \mathbb{R}_{>0}^l)$ there is exactly one tree $T \in \mathcal{T}_{k,l}^{root}$ which is positive at $(L; L')$ and such that $[\pi(T)] = c$.*

One can easily derive from Theorem 5.3 the statement of Theorem 3.2 in the case $g = 0$ and $W = H_{k,l}$: the number of trees $T \in \mathcal{E}_{0,k,l}^*$ positive at the point (L, L') is constant for $(L, L') \in H_{k,l}^\circ \cap (\mathbb{R}_{>0}^k \times \mathbb{R}_{>0}^l)$.

Indeed, there are clearly $(k+l-1)!$ equivalence classes in $\Pi_{k,l}$, so by Theorem 5.3 there are $(k+l-1)!$ *rooted* trees positive at any point $(L, L') \in H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$. Each *unrooted* tree positive at (L, L') gives rise to $k+l-1$ rooted positive trees. So the number of unrooted trees positive at (L, L') is equal to $(k+l-2)!$ (this was also proven before, see Lemma 3.10). In particular, it is constant.

In fact, the proof of Theorem 5.3, given in section 5.3.2, is a refinement of the proof of Theorem 3.2 in the case $g = 0$ (see section 3.2.1). The main idea is that one can flip the edges of a rooted tree in such a way as to keep the equivalence class of its prefix-postfix sequence constant.

5.2.2 Triangulations of the product of simplices

Using the prefix-postfix sequences of plane trees, we will define a family of triangulations of a certain polytope. First we set up some notation and recall some terminology.

Definition 5.4 (Triangulation)**.** A *triangulation* of a d-dimensional polytope P is a finite collection of d-dimensional simplices such that: their vertices are among the vertices of P, their union is P, intersection of any two simplices is a common (possibly empty) face of both.

For $k, l \geq 1$ let $\Delta_k \times \Delta_l$ be the polytope in $\mathbb{R}^k \times \mathbb{R}^l$ with coordinates $L_1, \ldots, L_k; L'_1, \ldots, L'_l$, defined by

$$L_1 + \ldots + L_k = L'_1 + \ldots + L'_l = 1, \;\; L_i \geq 0, \;\; L'_i \geq 0.$$

Clearly, it is the Cartesian product of two simplices $\Delta_k \subset \mathbb{R}^k$ and $\Delta_l \subset \mathbb{R}^l$ of dimensions $k-1$ and $l-1$ respectively, and so its dimension is equal to $k+l-2$.

For $1 \leq i \leq k, 1 \leq j \leq l$ denote by e_{ij} the point (vector) in $\mathbb{R}^k \times \mathbb{R}^l$ with $L_i = 1$, $L'_j = 1$ and all other coordinates being zero. The set

$$\{e_{ij}\}_{1 \leq i \leq k, 1 \leq j \leq l}$$

is then the set of vertices of the polytope $\Delta_k \times \Delta_l$.

For $T \in \mathcal{T}^{root}_{k,l}$, let Δ_T be the convex hull of the $k+l-1$ points e_{ij} such that there is an edge in T joining black vertex with label i and white vertex with label j. The following statement is well-known ([DLRS10, Lemma 6.2.8]). We give here a proof for completeness.

Lemma 5.5. *For any $T \in \mathcal{T}^{root}_{k,l}$, the polytope Δ_T is a simplex of dimension $k+l-2$. Conversely, any simplex of dimension $k+l-2$ with vertices in the set of vertices of $\Delta_k \times \Delta_l$ is equal to Δ_T for some $T \in \mathcal{T}^{root}_{k,l}$.*

Proof. Note that the linear combination of vectors e_{ij}, $i^\bullet j^\circ \in E(T)$, with weights w_{ij} is equal to the vector $\mathrm{vp}_T(w)$ of vertex perimeters of T equipped with a weight function w giving the edge $i^\bullet j^\circ$ the weight w_{ij}. Lemma 2.5 then says that e_{ij} span a linear subspace $H_{k,l}$ of dimension $k+l-1$. In particular, e_{ij} are affinely independent.

Conversely, consider a set S of $k+l-1$ points e_{ij}, and let G be the the graph on the set of vertices $\{1^\bullet, \dots, k^\bullet\} \cup \{1^\circ, \dots, l^\circ\}$ with edges $i^\bullet j^\circ$, $e_{ij} \in S$. If G is not a tree, it must be disconnected (since it has $k+l-1$ edges). But then the linear span of e_{ij} is included in the hyperplane $\sum_{i\in I} L_i = \sum_{j \in J} L'_j$, where I, J are the black and white labels in any fixed connected component of G. This hyperplane intersects $\Delta_k \times \Delta_l$ transversely, and so e_{ij} are not affinely independent. □

We are now ready to define the triangulations.

Theorem 5.6. *Let $k, l \geq 1$. For every equivalence class $c \in \Pi_{k,l}/\sim$ the simplices of the set*

$$\{\, \Delta_T \mid T \in \mathcal{T}^{root}_{k,l}, [\pi(T)] = c \,\}$$

form a triangulation of $\Delta_k \times \Delta_l$.

Recall the three conditions for a family of simplices to form a triangulation of the polytope (Definition 5.4). The vertices of Δ_T are the vertices of $\Delta_k \times \Delta_l$ be definition. The fact that the union of Δ_T covers $\Delta_k \times \Delta_l$ follows from Theorem 5.3. Indeed, the set of points of $\Delta_k \times \Delta_l$ at which a tree T is positive is exactly the interior of Δ_T. Thus Theorem 5.3 states that that the union of interiors of Δ_T is $\Delta_k \times \Delta_l$ minus its intersection with a finite number of hyperplanes, and it is enough to take the closure.

The only non-trivial part of the proof of Theorem 5.6 is to show that the intersections of the simplices Δ_T are proper. For this, we use a criterion due to Postnikov [Pos09], which characterizes pairs of trees T, T' such that Δ_T and $\Delta_{T'}$ intersect properly.

Recursive construction

Next, we show that these triangulations admit a recursive construction. For $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$, let $\Delta_k^{(i)} \times \Delta_l$ and $\Delta_k \times \Delta_l^{(j)}$ be the facets (i.e. faces of codimension 1) of $\Delta_k \times \Delta_l$ defined by $L_i = 0$ and $L'_j = 0$ respectively. Let also $\Delta_k^{(i)} \times \Delta_l^{(j)}$ be their intersection, a codimension 2 face of $\Delta_k \times \Delta_l$ defined by $L_i = L'_j = 0$. Clearly, these are also products of simplices of smaller dimensions. Their sets of vertices are (in order) $\{e_{i'j'}, i' \neq i\}$,

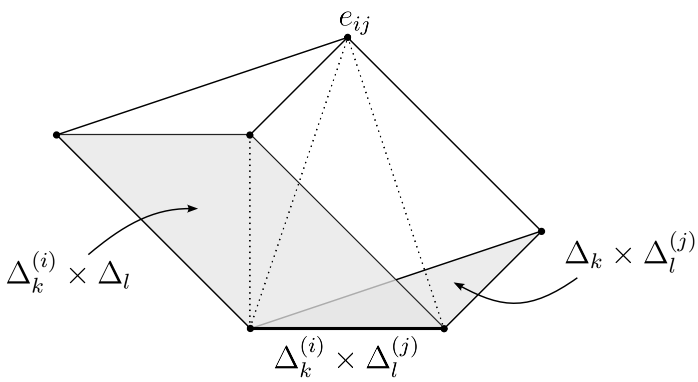


Figure 5.2: Recursive construction of triangulations (the particular case of a triangular prism $\Delta_2 \times \Delta_1$ is shown).

$\{e_{i'j'}, j' \neq j\}$ and $\{e_{i'j'}, i' \neq i, j' \neq j\}$. In particular, for each of these faces, one can construct a triangulation of it using Theorem 5.6, given a class of cyclic equivalence of sequences of labels in $\{1^\bullet, \ldots, k^\bullet\} \cup \{1^\circ, \ldots, l^\circ\}$ deprived of (respectively) $\{i^\bullet\}$, $\{j^\circ\}$ or $\{i^\bullet, j^\circ\}$.

For any any equivalence class $c \in \Pi_{k,l}/\sim$ denote by $c - i^\bullet$ the class of cyclic equivalence of a sequence π with $i^\bullet$ removed, where π is any representative of c. Define analogously $c - j^\circ$ and $c - i^\bullet - j^\circ$.

Finally, let Tr_c, $Tr_{c-i^\bullet}$, Tr_{c-j° and $Tr_{c-i^\bullet-j^\circ}$ be the triangulations of $\Delta_k \times \Delta_l$, $\Delta_k^{(i)} \times \Delta_l$, $\Delta_k \times \Delta_l^{(j)}$ and $\Delta_k^{(i)} \times \Delta_l^{(j)}$ respectively, corresponding (via Theorem 5.6) to c, $c - i^\bullet$, $c - j^\circ$ and $c - i^\bullet - j^\circ$ respectively.

Proposition 5.7. *Fix $k, l \geq 1$ and $c \in \Pi_{k,l}/\sim$. Take any two consecutive elements $i^\bullet$ and j° in c. Then:*

- *every simplex in Tr_c is the convex hull of e_{ij} and a simplex in either $Tr_{c-i^\bullet}$ or Tr_{c-j°; all simplices in Tr_c are obtained this way;*
- *Tr_c, $Tr_{c-i^\bullet}$ and Tr_{c-j° all induce the same triangulation of $\Delta_k^{(i)} \times \Delta_l^{(j)}$, namely $Tr_{c-i^\bullet-j^\circ}$.*

This recursive construction is schematically represented in Figure 5.2.

Count of triangulations

There are $(k + l - 1)!$ equivalence classes in $\Pi_{k,l}/\sim$, so Theorem 5.6 gives $(k+l-1)!$ triangulations of $\Delta_k \times \Delta_l$. However, not all of them are distinct. The reason for this is that: (a) the simplices on the vertex set of $\Delta_k \times \Delta_l$ naturally correspond to *non-plane* trees; (b) a tree can generally be embedded into the plane in several nonequivalent ways.

Nevertheless, we have the following conjectural formula for the number of *distinct* triangulations given by Theorem 5.6.

Conjecture 5.8. *For $k, l \geq 3$ and $(k, l) \neq (3, 3)$, the number of distinct triangulations in Theorem 5.6 is equal to $(k+l-1)! - \frac{3}{2}k!l!$. More precisely, $\frac{1}{2}k!l!$ triangulations are counted 4 times.*

We do not yet have a complete proof of this statement. Nevertheless, it is experimentally verified for the values of k, l with $k + l \leq 10$.

Note that the polytope $\Delta_k \times \Delta_l$ possesses a lot of symmetries. More precisely, the product of symmetric groups $S_k \times S_l$ acts naturally on the set of its vertices by permuting them. This action extends to an action on the whole polytope by Euclidean isometries. In particular, two triangulations of $\Delta_k \times \Delta_l$ are essentially the same if one can be obtained from the other by an action of such symmetry.

Conjecture 5.8 above implies that Theorem 5.6 gives at least $\frac{(k+l-1)!}{k!l!} - 1$ pairwise non-isometric triangulations of $\Delta_k \times \Delta_l$. In the case $k = l = n$ this is asymptotically exponential, since

$$\frac{(k+l-1)! - \frac{3}{2}k!l!}{k! \cdot l!} = \frac{1}{2n}\binom{2n}{n} - \frac{3}{2} \sim \frac{1}{2\sqrt{\pi}n^{3/2}}4^n, \; n \to \infty.$$

Bibliographic remarks

See [DLRS10, section 6.2] for an introduction to triangulations of products of simplices. Despite the simplicity of the polytope, the set of its triangulations is not completely understood. For example, describing the structure of the secondary polytope of $\Delta_k \times \Delta_l$ appears as an open problem in [GKZ94] and remains so to this day. Triangulations of this polytope, however, are interesting from several perspectives: theory of polytopes [BB98], toric varieties [San05], tropical geometry [DS04]. Only a couple of explicit constructions of triangulations of $\Delta_k \times \Delta_l$ are known (staircase triangulations [DLRS10, section 6.2] , Dyck path triangulations [CPS15]). On the enumeration side, only the asymptotics of their number is available [San05]. Recently, the triangulations of $\Delta_k \times \Delta_l$ were put in bijection with certain matching ensembles [OY13].

5.3 Proofs

5.3.1 Properties of prefix-postfix sequences

In this section we gather several elementary properties of the prefix-postfix sequences of trees which will be useful later.

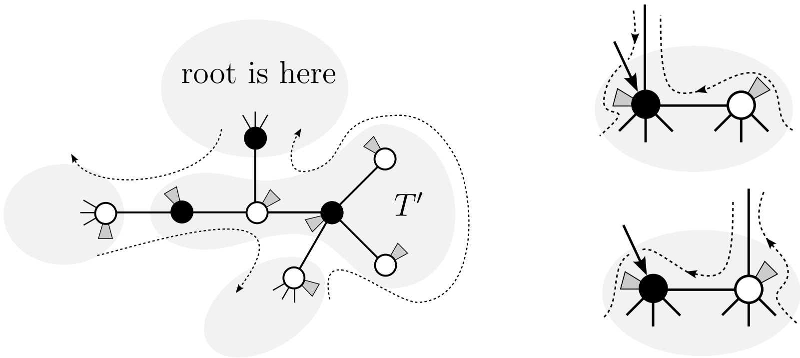


Figure 5.3: Left: going around T, we visit parts of the boundary of a subtree T' in the same order as just going around T'; the dotted lines represent the tour around T. Right: induced rooting of a subtree T'.

Definition 5.9 (Induced rooting of a subtree)**.** Let T' be a subtree of a tree T. The *rooting of* T' *induced by* T is defined by rooting T':

- at the corner of T' containing the root half-edge of T, if such a corner exists;
- otherwise, at the first visited black corner of T' when going around T (Figure 5.3, right).

Lemma 5.10.

1. *The first and the last element of $\pi(T)$ are the labels of the black and the white extremities of the root edge of T, respectively.*
2. *Let T' be a subtree of T. Then the order of vertex labels of T' in $\pi(T)$ coincides with $\pi(T')$, where T' has the rooting induced by T.*

Proof. The first claim follows directly from Definition 5.2. For the second claim, note that when going around T, we visit parts of the boundary of T' in the same order as if we were just going around T' (Figure 5.3, left). □

The proof of the next property will require a bit of case analysis.

Lemma 5.11. *Let T be any tree. Consider the prefix-postfix marking of T. Let c and c' be two consecutive marked corners, when going around T (hence c is not the last marked corner). Then the number of unmarked corners between c and c' is equal to:*

- *1, if c and c' are of the same color;*

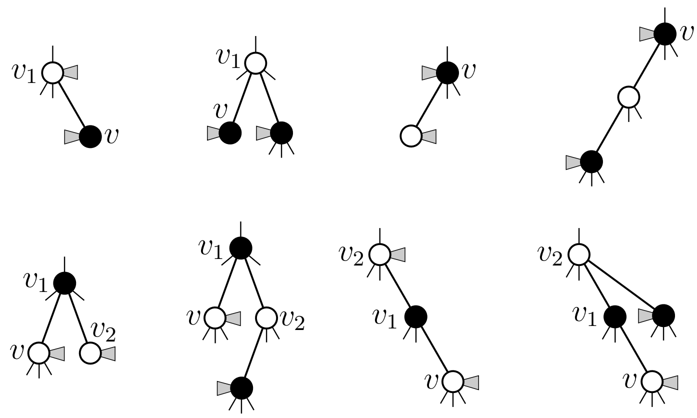


Figure 5.4: Possible configurations of two consecutive markers.

- *0, if c is black and c' is white;*
- *2, if c is white and c' is black.*

Proof. For any vertex v of T, define its *parent* as the first vertex on the path from v to *the white extremity of the root edge* (not the root vertex!). The *children* of v are all the vertices whose parent is v.

Let now v be the vertex of c. Since c is not the last marked corner, v is not the white extremity of the root edge. In particular, v has a parent v_1.

Suppose c is black. We distinguish 4 cases (see Figure 5.4, first row, from left to right).

Case 1: v is a leaf, v is the last child of v_1. In this case c' is the last corner of v_1, and there are no corners between c and c'.

Case 2: v is a leaf, v is not the last child of v_1. In this case c' is the first corner of the next child of v_1, and there is exactly one corner between c and c'.

Case 3: v is not a leaf, the first child of v is a leaf. In this case c' is the unique corner of this first child, and there are no corners between c and c'.

Case 4: v is not a leaf, the first child of v is not a leaf. In this case c' is the first corner of the first child of the first child of v, and there is exactly one corner between c and c'.

Now suppose c is white. We have 4 more cases to consider (see Figure 5.4, second row, from left to right).

Case 5: v is not the last child of v_1, the next child v_2 of v_1 is a leaf. In this case c' is the unique corner of v_2, and there is exactly one corner between c and c'.

Case 6: v is not the last child of v_1, the next child v_2 of v_1 is not a leaf. In this case c' is the first corner of the first child of v_2, and there are exactly two corners between c and c'.

Case 7: v is the last child of v_1. Let v_2 be the parent of v_1 (it exists since v_1 is black). We have two subcases.

Case 7a: v_1 is the last child of v_2. In this case c' is the last corner of v_2, and there is exactly one corner between c and c'.

Case 7b: v_1 is not the last child v_2. In this case c' is the first corner of the next child of v_2, and there is exactly two corners between c and c'. □

Lemma 5.12. *Let T be any tree and let $i^\bullet$ and j° be two consecutive labels in $\pi(T)$. Then either $i^\bullet$ is a leaf adjacent to j° or j° is a leaf adjacent to $i^\bullet$.*

Proof. This follows from the proof of Lemma 5.11. Indeed, in Figure 5.4 the cases of two consecutive markers, first being black and second being white, are the first and third in the first row. □

Lemma 5.13. *Let T be any tree and let $i^\bullet j^\circ \in E(T)$. Then $i^\bullet$ precedes j° in $\pi(T)$.*

Proof. For any vertex of T we define its parent and its children as in the proof of Lemma 5.11. If j° is a child of $i^\bullet$, then the first visit (and so the last visit as well) to j° is after the first visit to $i^\bullet$. If j° is a parent of $i^\bullet$ then the last visit to j° is after the first visit to $i^\bullet$. □

In the following lemma, by *embedding a tree in the plane* we mean embedding it as a ribbon graph, i.e. the orders of edges around each vertex in the embedding are those prescribed by the ribbon graph structure.

Lemma 5.14. *Any tree $T \in \mathcal{T}_{k,l}^{root}$ can be embedded in the plane in such a way that:*

- *the edges are straight line segments;*
- *the vertices lie on a circle;*
- *the counterclockwise cyclic order of vertices on the circle is given by $[\pi(T)]$.*

Proof. This is clear from topological considerations. Indeed, starting from any embedding of T in the plane, consider a path γ going counterclockwise around its boundary and close to it. The cyclic order of appearances of marked corners of T is given by $[\pi(T)]$. Now, continuously deform γ into a circle, "dragging along" the marked corners of T (see Figure 5.5).

A formal proof goes as follows. Start with a circle. Put the vertices on the circle in the cyclic order prescribed by $[\pi(T)]$. For every $i^\bullet j^\circ \in E(T)$,

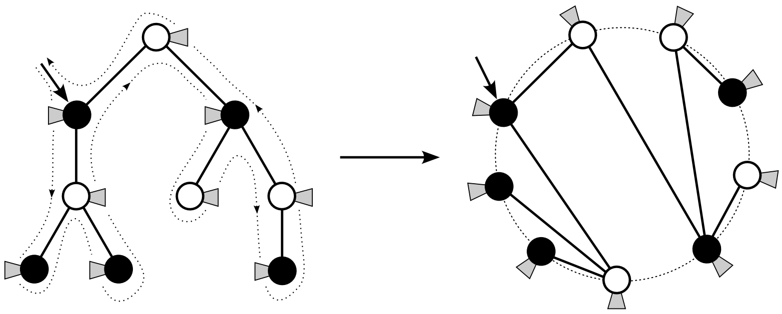


Figure 5.5: Embedding of a tree T with vertices on a circle. The counter-clockwise circular order of vertices on the circle is given by $[\pi(T)]$.

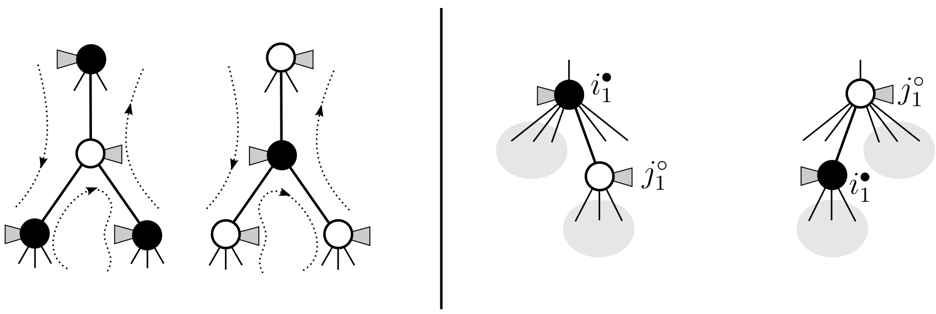


Figure 5.6: Proof of Lemma 5.14

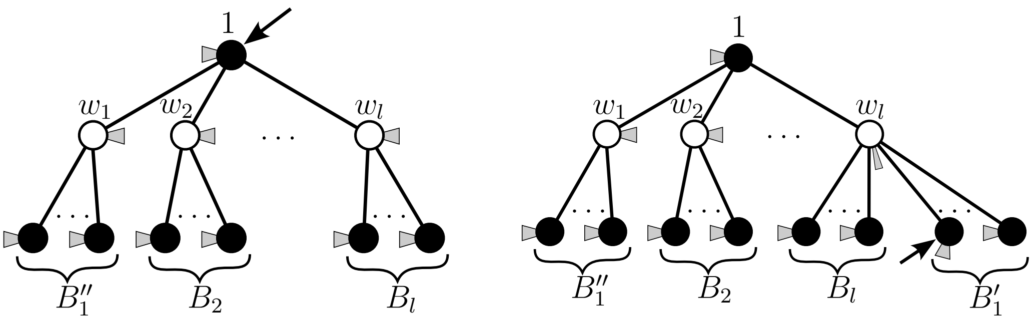


Figure 5.7: Trees T_0 positive at $(L; L')$ and satisfying $[\pi(T_0)] = c_0$.

draw a straight line segment connecting $i^\bullet$ and j°. It is enough to prove two claims. Firstly, for every vertex the circular order of segments adjacent to it is the same as in T. Secondly, no two segments intersect in their interiors.

The first claim is clear, since for any vertex in T the cyclic order of markers of all of its neighbors in $\pi(T)$ is the same as the cyclic order of corresponding edges around this vertex (see Figure 5.6, left).

To prove the second claim, assume two segments $i_1^\bullet j_1^\circ$ and $i_2^\bullet j_2^\circ$ intersect. By Lemma 5.13 their order in $\pi(T)$ is $i_1^\bullet, i_2^\bullet, j_1^\circ, j_2^\circ$. For any vertex of T we define its parent and its children as in the proof of Lemma 5.11. Then j_1° is either a child or a parent of $i_1^\bullet$. In both cases $i_2^\bullet$ is in one of the subtrees shaded gray in Figure 5.6, right. Then so is its neighbor j_2°. In particular, the last visit to j_2° is before the last visit to j_1°, a contradiction. □

5.3.2 Proof of Theorem 5.3

Lemma 5.15. *Let* $(L; L') = (N, 1, \ldots, 1; \frac{N+k-1}{l}, \ldots, \frac{N+k-1}{l}) \in \mathbb{R}^k \times \mathbb{R}^l$ *with* $N > kl$ *arbitrary. Then for every equivalence class* $c \in \Pi_{k,l}/\sim$ *there is exactly one tree* $T \in \mathcal{T}_{k,l}^{root}$ *which is positive at* $(L; L')$ *and such that* $[\pi(T)] = c$.

Proof. Lemma 3.10 shows that there are exactly $(k+l-2)!$ *non-rooted* trees that are positive at the point $(L; L')$. This makes $(k+l-1)!$ *rooted* trees positive at $(L; L')$. Since the number of equivalence classes in $\Pi_{k,l}/\sim$ is also $(k+l-1)!$, it is enough to prove that for each $c_0 \in \Pi_{k,l}/\sim$ there is *at least one* tree T_0 positive at $(L; L')$ and such that $[\pi(T_0)] = c$.

As shown in Lemma 3.10, the trees positive at $(L; L')$ are characterized as follows: all of the l white vertices are adjacent to the black vertex number 1; the remaining $k-1$ black vertices are attached to the white ones in an arbitrary manner (see Figure 5.7).

Now fix $c_0 \in \Pi_{k,l}/\sim$ and take the representative $\pi_0 \in \Pi_{k,l}$ of c_0 such that

$$\pi_0 = (B_1, w_1^\circ, B_2, w_2^\circ, \ldots, B_l, w_l^\circ),$$

where B_i are (possibly empty) blocks of consecutive black labels, $1^\bullet \in B_1$, and w_i° are the white labels. Let also $B_1 = (B_1', 1^\bullet, B_1'')$ where B_1', B_1'' are (possibly empty) blocks. There are two cases to consider.

Case 1: B_1' is empty. We define the tree T_0 by specifying the counter-clockwise circular orders of vertices adjacent to $1^\bullet, w_1^\circ, \ldots, w_l^\circ$ as follows (see Figure 5.7, left): the order around $1^\bullet$ is $(w_1^\circ, w_2^\circ, \ldots, w_l^\circ)$; for each $2 \le i \le l$, the order around w_i° is $(1^\bullet, B_i)$; the order around w_1° is $(1^\bullet, B_1'')$; the tree is rooted at the corner $w_l^\circ 1^\bullet w_1^\circ$. Direct inspection shows that $\pi(T_0) = \pi_0$.

Case 2: B_1' is non-empty. In this case the tree T_0 is defined as follows (see Figure 5.7, right): the order around $1^\bullet$ is $(w_1^\circ, w_2^\circ, \ldots, w_l^\circ)$; for each $2 \le i \le l-1$, the order around w_i° is $(1^\bullet, B_i)$; the order around w_1° is $(1^\bullet, B_1'')$; the order around w_l° is $(1^\bullet, B_l, B_1')$; the tree is rooted at the first vertex of B_1'. We then have $[\pi(T_0)] = [\pi_0] = c_0$. □

Let $T \in \mathcal{T}_{k,l}^{root}$ and let e be an edge of T not adjacent to a leaf. By *flipping* e we mean removing it from T and reconnecting the two created trees T_0, T_1 by a new edge e' in such a way that the black extremity of e' is in $T_{1-\varepsilon}$ if the black extremity of e was in T_ε. If e is the root edge, e' should also remain the root edge (hence rerooting is necessary). Otherwise, the root edge should not change.

Lemma 5.16. *For every tree $T \in \mathcal{T}_{k,l}^{root}$ and every edge e of T not adjacent to a leaf, there is one and only one way to flip e to get a tree $T' \in \mathcal{T}_{k,l}^{root}$ such that $[\pi(T)] = [\pi(T')]$. Moreover, if e is not the root edge of T, then $\pi(T) = \pi(T')$.*

Proof. Suppose the removal of e from T produces two trees T_0 and T_1, the black extremity of e being in T_0. We will distinguish several cases.

Case 1: e is the root edge of T. Using the second point of Lemma 5.10, we see that $\pi(T)$ is a concatenation of $\pi(T_0)$ and $\pi(T_1)$ rooted in such a way that their root edges are e_0 and e_1 respectively, where e_0, e_1 are as in Figure 5.8, top. After the flip, e' will still be the root, and so $\pi(T')$ will be the concatenation of $\pi(T_1)$ and $\pi(T_0)$ with possibly different rootings of these trees. But since $[\pi(T)] = [\pi(T')]$, the sequences $\pi(T_0)$ for the two rootings should be the same, and similarly for $\pi(T_1)$. The first point of Lemma 5.10 implies that the rootings should be the same. It means that the only possibility for e' is to be the edge connecting the corners c_0 and c_1 as in Figure 5.8, top. One easily checks that this choice is indeed valid.

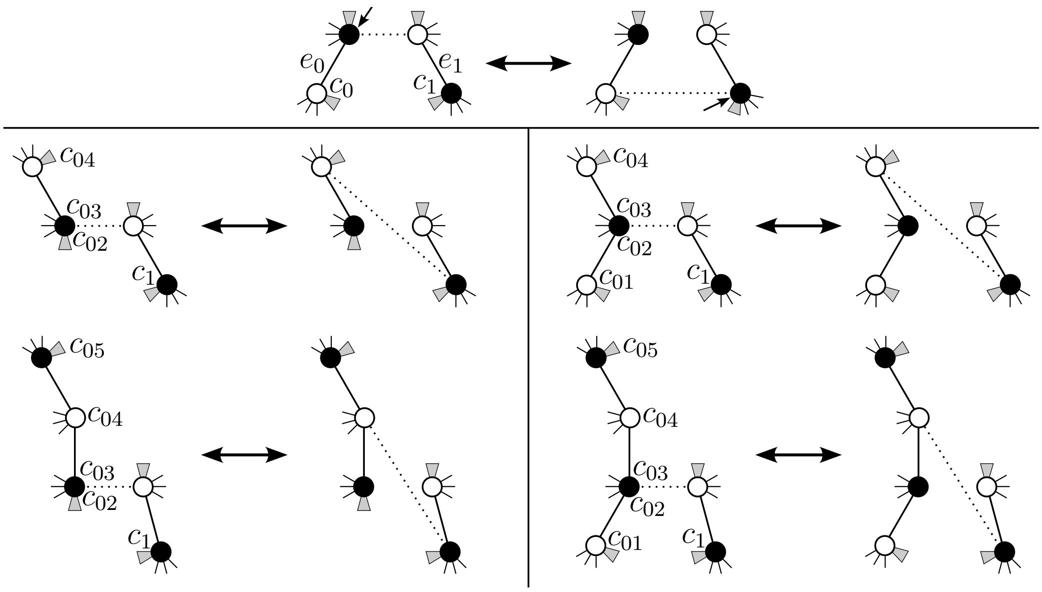


Figure 5.8: Flips preserving $[\pi(T)]$. In each case the dotted edges are e on the left and e' on the right.

Case 2: the root of T is in T_0. Since flipping e should not change the root, by the first point of Lemma 5.10 $\pi(T)$ and $\pi(T')$ should start and end with the same labels. Since $[\pi(T)] = [\pi(T')]$, we must have $\pi(T) = \pi(T')$.

Note that $\pi(T)$ is of the form $(A, \pi(T_1), B)$ with $(A, B) = \pi(T_0)$, where T_0 is rooted at the root of T and T_1 is rooted at the its first visited black corner when going around T. The equality $\pi(T) = \pi(T')$ means that: (a) the prefix-postfix sequences $\pi(T_1)$ with the rootings of T_1 induced by T and T' are the same; (b) in the tree T', the edge e' should touch the boundary of T_0 between the same two markers as does e in T. (a) implies that the two rootings of T_1 are the same, hence e' can only be glued to the first visited black corner c_1 of T_1 when going around T. We will now show that (b) implies that e' can only be glued to the white corner following (when going around T_0) the corner where e is glued.

Let c_{01}, c_{02} be the last white and black corners of T visited before reaching e when going around T. Let c_{03}, c_{04}, c_{05} be the first three corners of T visited after traversing e for the second time. Because the root of T is in T_0, the corner c_{02} is visited before the corner c_{03} when going along T. In particular, the corner c_{03} is not marked.

If c_{02} is marked, by Lemma 5.11 either c_{04} or c_{05} are marked (Figure 5.8, bottom left). In both cases (b) implies that e' must be glued to c_{04}.

If c_{02} is not marked, by Lemma 5.11 either c_{01} and c_{04} are marked, or

c_{01} and c_{05} are marked (Figure 5.8, bottom right). In both cases (b) implies that e' can only be glued to either c_{01} or c_{04}. But gluing to c_{01} would force the label of the corresponding vertex to appear after $\pi(T_1)$ in $\pi(T')$, thus implying $\pi(T') \neq \pi(T)$. Hence, again, the only possibility we are left with is to glue e' to c_{04}.

One easily checks that, in all four cases, gluing e' to c_{04} and c_1 does produce a tree T' with $\pi(T) = \pi(T')$.

Case 3: the root of T is in T_1. This case is symmetric to the previous one. The corresponding flip is the inverse of the flip in Case 2 (one should read each case in Figure 5.8 from right to left). □

Proof of Theorem 5.3. The proof is a refinement of the proof of Theorem 3.2 in the case $g = 0$, so we only explain the general strategy. Refer to section 3.2.1 for details.

Lemma 5.15 states that the theorem is true for a specific point $(L; L')$. When $(L; L')$ varies inside a connected component of $H^{\circ}_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$ every tree T remains either positive at $(L; L')$ or non-positive at $(L; L')$ So it is enough to prove that, when $(L; L')$ traverses a wall, one can establish a bijection between the trees that cease to be positive at $(L; L')$ and those that become positive at $(L; L')$, and this bijection should preserve the equivalence class of the prefix-postfix sequence of a tree.

When $(L; L')$ traverses a wall at a point not belonging to other walls, a tree can cease to be positive at $(L; L')$ only because *exactly one* of its edges (not adjacent to a leaf) becomes zero-length (the length of such edge is given by the linear form defining the wall). As explained in section 3.2.1, flipping this edge (in any possible way) produces a tree which becomes positive at $(L; L')$ after the traversal of the wall. By Lemma 5.16 this flip can be chosen in such a way as to preserve the equivalence class of the prefix-postfix sequence of a tree, and this choice is unique. The bijection is thus defined by doing this particular flip on the zero-length edge. □

5.3.3 Triangulations are well-defined

In this section we prove Theorem 5.6. As explained right after its statement, it is enough to check that for any to simplices Δ_T and $\Delta_{T'}$, their intersection is a common face of both. To do this, we will use the following criterion due to Postnikov.

Lemma 5.17 ([Pos09], Lemma 12.6)**.** *Let $U(T, T')$ be the union of edges of T and T', with the edges of T oriented from their black to their white extremities, and the edges of T' oriented from their white to their black extremities.*

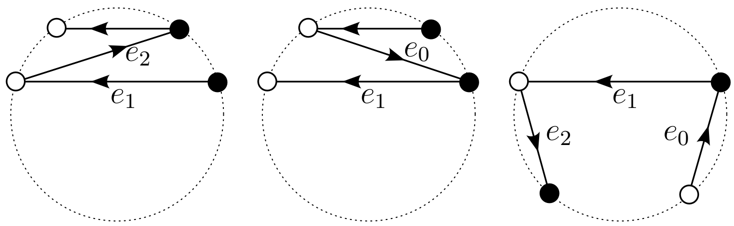


Figure 5.9

Then $\Delta_T \cap \Delta'_T$ is a common face of both simplices if and only if $U(T, T')$ does not contain a simple directed cycle of length at least 4.

Proof of Theorem 5.6. Assume, for the sake of contradiction, that there are two trees $T_1, T_2 \in \mathcal{T}_{k,l}^{root}$ with $[\pi(T_1)] = [\pi(T_2)] = c$ and such that $U(T_1, T_2)$ has a simple directed cycle γ of length at least 4.

By Lemma 5.14, it is possible to embed T_1 and T_2 simultaneously in the plane so that their vertices coincide and lie on a circle, the order of vertices counterclockwise is given by c, and the edges are straight line segments.

Let $e_1 \in \gamma$ be the edge in γ with the property that it is the edge of T_1 with the least vertices to the right of it (when oriented from black to white). If there are several such edges, take any one of them. Note that, by Lemma 5.13, the root edge of T_1 is to the left of e_1.

Let e_0, e_2 be the previous and the next edges in γ. We claim that their other extremities are to the left of e_1. Indeed, if the other extremity of e_2 were to the right of e_1, the next edge in γ after e_2 would be an edge of T_1 which does not intersect e_1 (since T_1 is embedded) and has the root edge of T_1 to the left of it (again by Lemma 5.13), see Figure 5.9, left. But such edge would have less vertices to the right of it then e_1, a contradiction. The proof for the other extremity of e_0 is analogous (Figure 5.9, center).

Thus the other extremities of e_0 and e_2 are to the left of e_1. Since T_2 is embedded, e_0 and e_2 do not intersect, and the configuration of segments is as in Figure 5.9, right. But then, by Lemma 5.13, the root edge of T_2 should be both to the right of e_0 and to the right of e_2 (when oriented from white to black), which is impossible. □

5.3.4 Recursive construction of triangulations

Proof of Proposition 5.7. First note that for any $T \in \mathcal{T}_{k,l}^{root}$, $\Delta_T \cap \{L_i = 0\}$ is a face of Δ_T of codimension $\deg_T(i^\bullet)$. Indeed, imposing $L_i = 0$ forces the $\deg_T(i^\bullet)$ edges adjacent to $i^\bullet$ to have zero weight. Analogously, $\Delta_T \cap \{L'_j = 0\}$ is a face of Δ_T of codimension $\deg_T(j^\circ)$.

Let now $T \in \mathcal{T}_{k,l}^{root}$ be a tree with $[\pi(T)] = c$. By Lemma 5.12, either $i^\bullet$ is a leaf adjacent to j° or j° is a leaf adjacent to $i^\bullet$. In the first case, $\Delta_T \cap \{L_i = 0\}$ is a codimension 1 face of Δ_T and is a simplex of the triangulation $Tr_{c-i^\bullet}$ of $\Delta_k^{(i)} \times \Delta_l$ corresponding to the tree T with the leaf $i^\bullet$ removed. In the second case, $\Delta_T \cap \{L'_j = 0\}$ is a codimension 1 face of Δ_T and is a simplex of the triangulation Tr_{c-j° of $\Delta_k \times \Delta_l^{(j)}$ corresponding to the tree T with the leaf j° removed.

Conversely, starting from any simplex in $Tr_{c-i^\bullet}$ or Tr_{c-j° and the corresponding tree T', there is exactly one way to join the missing vertex to T' in such a way that the resulting prefix-postfix sequence is in the equivalence class c (if $i^\bullet$ is missing, it should be glued to j° right before its marker; if j° is missing, it should be glued to $i^\bullet$ right after its marker). This finishes the proof of the first claim.

To prove the second claim, it is enough to show that Tr_c induces the triangulation $Tr_{c-i^\bullet-j^\circ}$ on $\Delta_k^{(i)} \times \Delta_l^{(j)}$. Suppose a tree $T \in \mathcal{T}_{k,l}^{root}$ is such that $\Delta_T \cap \{L_i = L'_j = 0\}$ is a codimension 2 face of Δ_T (and so is a simplex of the induced triangulation on $\Delta_k^{(i)} \times \Delta_l^{(j)}$). Then either $i^\bullet$ is a leaf and j° is of degree 2, or the opposite (otherwise imposing $L_i = L'_j = 0$ would force more than two edges to become zero-weight). In both cases the codimension 2 face is a simplex of $Tr_{c-i^\bullet-j^\circ}$ corresponding to the tree T with both $i^\bullet$ and j° removed. $\square$

Chapter 6

Many-vertex graphs

In this chapter we study the counting functions for face-bipartite metric ribbon graphs with more than one vertex. Although their top-degree terms are not polynomial, we present a conjecture (Conjecture 6.4) that the top-degree term outside of the walls becomes polynomial if the contribution of each graph is weighted by the count of certain spanning trees (t-*weight*, see Definition 6.2). We also give a conjectural explicit expression for this top-degree term. We prove the conjecture in the genus zero case (Theorem 6.5). The two main tools of the proof are Proposition 5.3 about the count of metric plane trees with given prefix-postfix sequence, and a bijection due to Bernardi between plane maps with a distinguished spanning tree and pairs of plane trees.

In section 6.1 we define the t-weights, state Conjecture 6.4 and Theorem 6.5, and explain a particular case of the conjecture which is easy to prove. In section 6.2 we explain the strategy of proof of Theorem 6.5 and introduce the necessary tools. Section 6.3 contains the proofs.

6.1 Introduction

6.1.1 Main Conjecture and Theorem

Let $g \geq 0$, $k, l \geq 1$ and $n \geq 1$. Let also $d = (d_1, \dots, d_n)$ be a composition of legnth n. Recall that $\mathcal{RG}^{2d}_{g,(k,l)}$ denotes the set of isomorphism classes of genus g ribbon graphs which are face-bicolored, with k black boundary components labeled from 1 to k, l white boundary components labeled from 1 to l, and with n vertices labeled from 1 to n, of degrees $2d_1, \dots, 2d_n$ respectively.

The set $\mathcal{RG}^{2d}_{g,(k,l)}$ is non-empty if and only if the following condition is satisfied (it is equivalent to the Euler's relation):

$$d_1 + \ldots + d_n = k + l + n + 2g - 2.$$

Let us first consider the counting functions $\mathcal{P}^{2d}_{g,(k,l)}(L, L')$ of the family $\mathcal{RG}^{2d}_{g,(k,l)}$. Their general behaviour follows from Proposition 2.16.

Proposition 6.1. *Let C be an open cell in $\mathcal{PS}_{k,l}$. Then for $(L, L') \in C \cap (\mathbb{Z}^k \times \mathbb{Z}^l)$, the function $\mathcal{P}^{2d}_{g,(k,l)}(L, L')$ is either identically zero or a polynomial of degree $2g + n - 1$.*

Proof. Any connected component C of $H^+_{k,l} \cap W^\circ$ is an open cell of the polyhedral subdivision $\mathcal{PS}_{k,l}$ of $H_{k,l}$. Thus, by Proposition 2.16, the contribution of any $G \in \mathcal{RG}^{2d,*}_{g,(k,l)}$ to $\mathcal{P}^{2d}_{g,(k,l)}$ on C is either zero or a polynomial of degree

$$|E(G)| - |V(G)| + 1 = |F(G)| + (2g - 2) + 1 = 2g + n - 1.$$

□

After Theorem 3.2 about the polynomiality of the top-degree term of the counting functions for one-vertex graphs, one would hope that the same phenomenon occurs for many-vertex graphs. However, this is no longer the case. The minimal counter-example is $g = 0$, $(k, l) = (2, 2)$, $n = 2$ and $2d = (2, 2)$. As in section 3.1, we can compute $\mathcal{P}^{(2,2)}_{0,(2,2)}(L, L')$ on the cells $C_1 = \{L_1 < L'_1, L_1 < L'_2\}$ and $C_2 = \{L_1 > L'_1, L_1 < L'_2\}$ to get

$$\mathcal{P}^{(2,2)}_{0,(2,2)}|_{C_1}(L, L') = 2(L_2 - 3),$$
$$\mathcal{P}^{(2,2)}_{0,(2,2)}|_{C_2}(L, L') = 2(L'_2 - 3).$$

Their difference $2(L_2 - L'_2)$ has degree 1 and is not divisible by $L_1 + L_2 - L'_1 - L'_2$ (which is the defining equation of the hyperplane $H_{2,2}$), and so the top-degree terms are indeed different.

Nevertheless, we conjecture that by properly weighting the contribution of each graph, one can recover the polynomiality.

Definition 6.2 (t-weight)**.** For any $G \in \mathcal{RG}_{g,(k,l)}$ define $t(G)$, the *t-weight* of G, as follows:

- orient each edge e of G in such a way that the black boundary adjacent to e is on the left of e; call the obtained oriented graph $\overrightarrow{G}$;
- $t(G)$ is equal to the number of spanning trees of $\overrightarrow{G}$ all of whose edges are oriented away from an (arbitrary) fixed vertex v of $\overrightarrow{G}$.

Lemma 6.3. *The t-weight of a ribbon graph $G \in \mathcal{RG}_{g,(k,l)}$ is well-defined, i.e. it does not depend on the choice of the vertex v.*

Proof. Since the ribbon graph G is face-bicolored, the colors of corners around each vertex alternate. Consequently, each vertex of the oriented graph $\overrightarrow{G}$ has equal in- and out-degrees. It is known that the number of oriented spanning trees of such oriented graphs does not depend on the choice of the source vertex, see for example [Sta18, Corollary 10.3]. □

We can now introduce the following weighted counting function:

$$\widetilde{\mathcal{P}}^{2d}_{g,(k,l)}(L,L') = \sum_{G \in \mathcal{RG}^{2d}_{g,(k,l)}} \frac{t(G)}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L,L').$$

The following is our main conjecture.

Conjecture 6.4. *Let $g \geq 0$, $k,l,n \geq 1$ and let d be a composition of $k+l+n+2g-2$ of length n. Then for every connected component C of $H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$, we have*

$$top_C\left(\widetilde{\mathcal{P}}^{2d}_{g,(k,l)}\right)(L,L') = (L_1 + \ldots + L_k)^{n-1} \cdot P^g_{H_{k,l}}(L,L'),$$

where $P^g_{H_{k,l}}$ is the polynomial giving the top-degree term on $H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$ of the counting function $\mathcal{P}^g_{k,l}$ for one-vertex graphs (see Proposition 3.3).

Note the that the expression for the top-degree term of $\widetilde{\mathcal{P}}^{2d}_{g,(k,l)}$ is independent of $2d$, the vector of degrees of the faces.

In what follows, we will prove Conjecture 6.4 in the particular case $g = 0$.

Theorem 6.5. *Let $k,l,n \geq 1$ and let d be a composition of $k+l+n-2$ of length n. Then for every connected component C of $H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$, we have*

$$top_C\left(\widetilde{\mathcal{P}}^{2d}_{0,(k,l)}\right)(L,L') = (k+l-2)! \cdot (L_1 + \ldots + L_k)^{n-1}.$$

We were able to check the conjecture for $g \leq 2, k,l,n \leq 3$, by calculating (on computer) the top-degree terms of $\widetilde{\mathcal{P}}^{2d}_{g,(k,l)}$ on different connected components of $H^\circ_{k,l}$. The complexity of the computation increases rapidly, which makes checking for larger values of g,k,l,n prohibitive.

Figure 6.1: Bijection between valid spanning trees of $\overrightarrow{G}$ and $\overrightarrow{G'}$ (dashed edges do not belong to the trees).

6.1.2 A particular case

There is one case of Conjecture 6.4 which is easy to prove. It is the case when $d = (k+l+2g-1, 1, 1, \dots, 1)$. This is explained by the following elementary observation.

Lemma 6.6. *Let $g \geq 0$, $k, l, n \geq 1$, let $d = (d_1, \dots, d_n)$ be a composition of $k+l+n+2g-2$ of length n. Let also $(L, L') \in H_{k,l} \cap (\mathbb{Z}^k_{>0} \times \mathbb{Z}^l_{>0})$. Then*

$$\widetilde{\mathcal{P}}^{(2d_1,\dots,2d_n,2)}_{g,(k,l)}(L, L') = \left(\sum_{i=1}^{n} L_i - (k+l+n+2g-2) \right) \cdot \widetilde{\mathcal{P}}^{(2d_1,\dots,2d_n)}_{g,(k,l)}(L, L').$$

Proof. Let $G \in \mathcal{RG}^{(2d_1,\dots,2d_n,2)}_{g,(k,l)}$ and let w be an integer metric on G with boundary perimeters (L, L'). Denote by $G' \in \mathcal{RG}^{(2d_1,\dots,2d_n)}_{g,(k,l)}$ the ribbon graph obtained from G by removing the vertex v_{n+1} with label $n+1$ and merging the two incident edges into one. w gives rise to a metric w' on G' with the same boundary perimeters, if we assign to the newly created edge of G' the length equal to the sum of lengths of the merged edges of G'.

Conversely, to reconstruct (G, w) from (G', w'), we need the additional data of the position of the vertex v_{n+1}. This data is the choice of an edge of G', and the choice of an ordered partition of the length of this edge into two positive integers. The total number of such choices is $\sum_{e \in E(G')}(w'(e) - 1) = \sum_{i=1}^{n} L_i - |E(G')| = \sum_{i=1}^{n} L_i - (k+l+n+2g-2)$.

Finally, we have to check that the t-weights (see Definition 6.2) of G and G' coincide. For this, we establish a bijection between directed spanning trees of $\overrightarrow{G}$ and $\overrightarrow{G'}$ whose edges are directed away from the vertex with label $1 \neq n+1$. Call such trees *valid*. Let T be a valid spanning tree of $\overrightarrow{G}$. If v_{n+1} is a leaf of T, delete it and the incident edge from T to obtain a valid spanning tree T' of $\overrightarrow{G'}$ (Figure 6.1, left). If v_{n+1} has degree 2 in T, replace the two incident edges by one oriented edge going from the origin vertex of the edge entering v_{n+1} to the end vertex of the edge exiting v_{n+1} (Figure 6.1, right). This also gives a valid spanning tree T' of $\overrightarrow{G'}$. It is clear that the correspondence between T and T' is bijective. $\square$

Corollary 6.7. *Let $g \geq 0$, $k, l, n \geq 1$ and let $d_0 = (k+l+2g-1, 1, 1, \ldots, 1)$ be the unique (up to permutation) composition of $k+l+n+2g-2$ of length n. Then for every connected component C of $H^{\circ}_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$, we have*

$$top_C\left(\widetilde{\mathcal{P}}^{2d_0}_{g,(k,l)}\right)(L, L') = (L_1 + \ldots + L_k)^{n-1} \cdot P^g_{H_{k,l}}(L, L').$$

Proof. Induction on n. The base case $n = 1$ is true by definition of $P^g_{H_{k,l}}$ and the fact that one-vertex graphs have only one spanning tree. The induction step follows from Lemma 6.6 by restricting to the top-degree term. $\square$

Unfortunately, we do not have a direct explanation for why the top-degree terms of $\widetilde{\mathcal{P}}^{2d}_{g,(k,l)}$ are independent of d. Combined with Corollary 6.7, such explanation would yield a proof of Conjecture 6.4. The proof of Theorem 6.5 below shows this independence in the case $g = 0$ in a slightly indirect way.

6.2 Strategy of proof and tools

6.2.1 Passing to the dual and rooting

Passing to the dual

First of all, we would like to redefine the weighted counting functions $\widetilde{\mathcal{P}}^{2d}_{0,(k,l)}$ using dual graphs.

Recall from section 1.1.2, that $\mathcal{RG}^{2d,*}_{g,(k,l)}$ denotes the set of isomorphism classes of genus g ribbon graphs which are vertex-bicolored, with k black vertices labeled from 1 to k, l white vertices labeled from 1 to l, and with n boundary components labeled from 1 to n, of degrees $2d_1, \ldots, 2d_n$ respectively.

For $G \in \mathcal{RG}^{2d}_{0,(k,l)}$, we need to reinterpret its t-weight $t(G)$ in terms of its dual $G^* \in \mathcal{RG}^{2d,*}_{0,(k,l)}$.

Lemma 6.8. *Let $G \in \mathcal{RG}^{2d}_{0,(k,l)}$ and let T be a spanning tree of $\overrightarrow{G}$ whose edges are oriented away from the vertex with label i. Then the set of edges $\{e^* \in E(G^*) : e \notin E(T)\}$ forms a spanning tree T^* of G^* with the following property: if one goes around T^* (crossing edges in $E(G^*) \setminus E(T^*)$) starting at a corner of the face of G^* with label i, then for any edge $e^* \in E(G^*) \setminus E(T^*)$ we cross it for the first time when visiting a black corner of T^*, see Figure 6.2.*

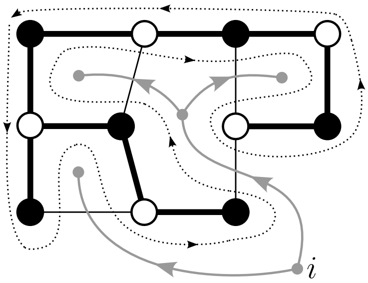


Figure 6.2: Dual graph G^* (solid black), oriented spanning tree T of $\overrightarrow{G}$ whose edges are oriented away from vertex i (grey), corresponding spanning tree T^* of G^* (thick edges of G^*); going around T^* is represented by a dotted line.

Moreover, the correspondence $T \leftrightarrow T^*$ is a bijection between the spanning trees of $\overrightarrow{G}$ and G^* with these properties.

The proof of this lemma is postponed to section 6.3.1.

Lemma 6.8 allows us to define the t-weights of graphs in $\mathcal{RG}^{2d,*}_{0,(k,l)}$.

Definition 6.9 (t-weight, dual planar version)**.** Let $G \in \mathcal{RG}^{2d,*}_{0,(k,l)}$. The *$t$-weight* of G is the number of spanning trees T of G with the property that, when going around T (crossing the edges in $E(G) \setminus E(T)$) starting in a corner of the face of G with (arbitrary fixed) label i, every edge in $E(G) \setminus E(T)$ is first crossed when visiting a black corner of T.

Finally, we have the alternative definition

$$\widetilde{\mathcal{P}}^{2d}_{0,(k,l)}(L, L') = \sum_{G \in \mathcal{RG}^{2d,*}_{0,(k,l)}} \frac{t(G)}{|\operatorname{Aut}(G)|} \cdot \mathcal{P}_G(L, L'),$$

where (recall Convention 1.7) $\mathcal{P}_G$ counts the number of metrics on G with *vertex* perimeters (L, L'). In the rest of this chapter we will exclusively use this alternative definition.

Rooting

In what follows, the ribbon graphs will always be rooted. Recall from section 1.1.1 that a rooting of a vertex-bicolored ribbon graph G is a choice of a black corner of G, denoted by an oriented half-edge pointing to that corner. Recall also the definitions of the root vertex, the root edge and the root face. The set of rooted ribbon graphs whose underlying unrooted ribbon graph is in $\mathcal{RG}^{2d,*}_{0,(k,l)}$ is denoted by $\mathcal{RG}^{2d,*,root}_{0,(k,l)}$.

6.2.2 Strategy of proof

We are now ready to explain the strategy of proof of Theorem 6.5.

Let $G \in \mathcal{RG}^{2d,*,root}_{0,(k,l)}$ be such that the root face has a fixed label $i_{root} \in \{1,\dots,n\}$. Then $t(G)$ is equal to the number of spanning trees of G satisfying the condition of Definition 6.9 with $i = i_{root}$. Let T be any such spanning tree of G. Suppose also that G is equipped with a metric w with vertex perimeters $(L, L') \in H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$.

Denote by $A^d_{k,l,i_{root}}(L, L')$ the set of such triples (G, T, w). $A^d_{k,l,i_{root}}(L, L')$ can be identified with the disjoint union of interiors of polytopes:

$$A^d_{k,l,i_{root}}(L, L') = \bigsqcup_{G,T} \operatorname{int} M_G(L, L'),$$

where the union is over $G \in \mathcal{RG}^{2d,*,root}_{0,(k,l)}$ which are rooted at a face with label i_{root}, are positive at (L, L'), and T is a spanning tree of G as above.

Call the *volume of* $A^d_{k,l,i_{root}}(L, L')$ the sum of volumes of all the $M_G(L, L')$. Let C be the cell of $\mathcal{PS}_{k,l}$ containing the point (L, L'). Then by Lemma 2.22 the volume of $A^d_{k,l,i_{root}}(L, L')$ coincides with

$$d_{i_{root}} \cdot top_C \left(\widetilde{\mathcal{P}}^{2d}_{0,(k,l)}\right)(L, L'), \tag{6.1}$$

where the extra term $d_{i_{root}}$ comes from the rooting (each graph in $\mathcal{RG}^{2d,*}_{0,(k,l)}$ has $d_{i_{root}}$ black corners in the face with label i_{root}).

To compute the volumes of the sets $A^d_{k,l,i_{root}}(L, L')$, we will first construct (using the Bernardi bijection described in section 6.2.3) a map ψ from $A^d_{k,l,i_{root}}(L, L')$ to a certain set $B^{d'}_{k,l}(L)$ of decorated plane trees (note that it is independent of L'), see section 6.3.2. This set will be identified with a disjoint union of products of simplices and the map ψ will be proved to be affine, volume-preserving and injective.

The images in $B^{d'}_{k,l}(L)$ of the maps ψ for several different values of d and i_{root} will form a partition of (a full measure subset of) $B^{d'}_{k,l}(L)$ (Proposition 6.16). Combining this with an explicit expression for the volume of $B^{d'}_{k,l}(L)$ (Lemma 6.12), we will get a linear relation between the volumes of $A^d_{k,l,i_{root}}(L, L')$ (Corollary 6.17). This relation allows to prove Theorem 6.5 by a simple induction.

The central claim of the proof is Proposition 6.16. Its proof relies on the construction of the Bernardi bijection and on Theorem 5.3 about the count of metric plane trees with given prefix-postfix sequence.

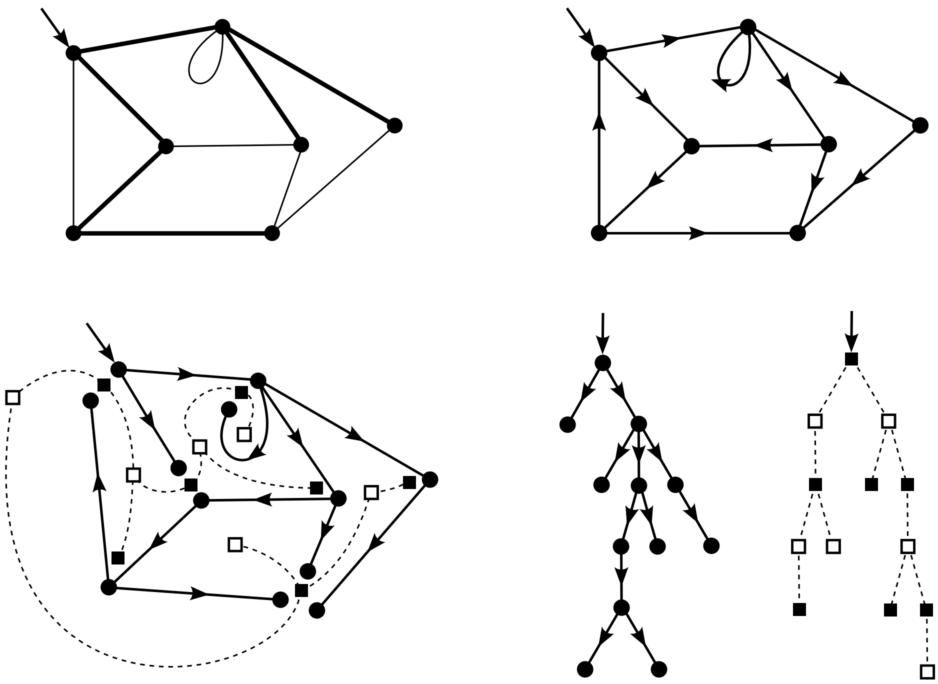


Figure 6.3: *Top left*: ribbon graph G with a distinguished spanning tree T (thick edges). *Top right*: corresponding orientation of G. *Bottom left*: construction of G' and G''. *Bottom right*: G' and G'' are trees.

6.2.3 Bernardi bijection

The main tool of the proof of Theorem 6.5 is a bijection of Bernardi [Ber07] between rooted plane maps with a distinguished spanning tree and pairs of rooted plane trees. It was generalized in [BC11] to "covered maps" of arbitrary genus. Although we only need the bijection for plane maps, we will use some of the formalism from [BC11] (one of the resulting trees is vertex-bicolored; the reconstruction procedure, see below).

In this section we describe the original bijection. How it is applied in our particular case is described in the next sections. We omit the technical details of the constructions, see [Ber07], [BC11].

Let G be a rooted plane ribbon graph (no labels or bicoloring), and let T be a spanning tree of G (Figure 6.3, top left). The tree T is considered to be rooted at the corner containing the root half-edge of G. We now construct a pair of rooted plane ribbon graphs (G', G''), with G'' vertex bicolored.

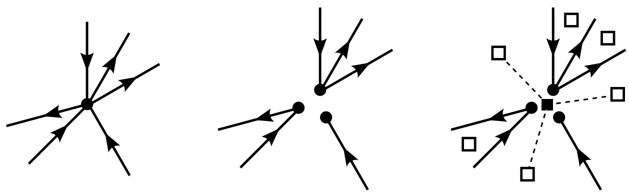


Figure 6.4: *Left and center*: exploding a vertex of G to construct G'. *Right*: local rule for constructing G'' (vertices are squares, edges are dashed).

Construction of G'

Orient the edges of G in the following way. First orient the edges of T away from the root vertex. Orient all other edges of G in such a way that, when going around T (crossing the edges in $E(G) \setminus E(T)$) starting at the root half-edge, every edge in $E(G) \setminus E(T)$ is first crossed at its end vertex. Such orientation is clearly unique (Figure 6.3, top right).

Now, "explode" each vertex of G as in Figure 6.4, left and center. Formally, let $v \in V(G)$ and let $(e_1, \ldots, e_{\deg_G(v)})$ be the edges incident to v written in (some) clockwise order. Suppose v has indegree n for the orientation constructed above and the incoming edges are $e_{i_1}, \ldots, e_{i_n}$ with $i_1 < \ldots < i_n$. Then the vertex v gets replaced by n vertices with clockwise orders of edges being $(e_{i_1}, \ldots, e_{i_2-1})$, $(e_{i_2}, \ldots, e_{i_3-1})$, …, $(e_{i_n}, \ldots, e_{i_1-1})$. Note that the root-half edge should be counted as an incoming edge of the root vertex. Also note that, if $n = 1$, there is no "explosion" per se – the vertex is not modified.

G' is the obtained plane rooted ribbon graph (Figure 6.3, bottom left). Its rooting is determined by the position of the root half-edge.

Construction of G''

The vertices of G'' correspond to the groups of exploded vertices (hence to the vertices of G) and to the faces of G. We color the first ones in black and the second ones in white and denote them by boxes rather than circles. The edges incident to a black vertex of G'' and their order are determined by the local rule given in Figure 6.4, right. Namely, there is an edge of G'' before each incoming edge of G' (when going clockwise around the corresponding vertex). See Figure 6.3, bottom left. G'' is rooted at the corner containing the root half-edge of G.

The bijection

Theorem 6.10 ([Ber07])**.** *The resulting ribbon graphs G' and G'' are trees. The edges of G' are oriented away from its root vertex (Figure 6.3, bottom*

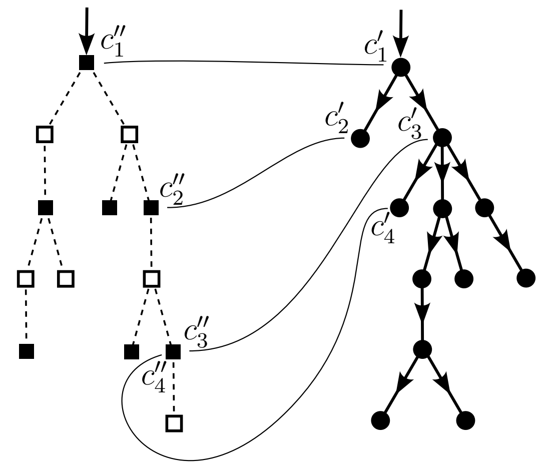


Figure 6.5: Gluing corners of G' and G'' to reconstruct G.

right).

The map $(G, T) \mapsto (G', G'')$ establishes a bijection between rooted plane maps with N edges and a distinguished spanning tree, and pairs consisting of a rooted plane tree with N edges and a rooted vertex-bicolored plane tree with $N + 1$ edges.

We now explain how to recover G from G' and G''. We do not precise how to reconstruct T, since we will not need this in our proof.

To reconstruct G from (G', G''):

- go counterclockwise around G' starting at its root half-edge; let $c'_1, \ldots, c'_{N+1}$ be the corners of first visits to the vertices of G' (in order of visit);
- go clockwise around G'' starting at its root half-edge; let $c''_1, \ldots, c''_{N+1}$ be all the visited black corners of G'' (in order of visit);
- for each $i = 1, \ldots, N + 1$, glue the corners c'_i and c''_i (i.e. glue the corresponding vertices along these corners);
- remove from the glued graph the edges of G'', the white vertices of G'' and forget the orientations of edges of G';
- the resulting graph is G.

See Figure 6.5 for an illustration. This procedure can be seen as "gluing T' to itself along T''". We say that the corners c'_i and c''_i are *corresponding corners of T' and T''.*

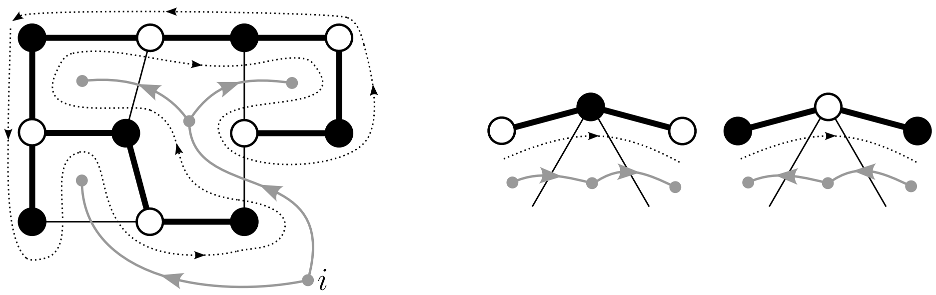


Figure 6.6: *Left*: dual graph G^* (solid black), oriented spanning tree T of $\overrightarrow{G}$ whose edges are oriented away from vertex i (grey), corresponding spanning tree T^* of G^* (thick edges of G^*); going around T^* is represented by a dotted line. *Right*: local picture of going around T^* (and T).

6.3 Proofs

6.3.1 Proof of Lemma 6.8

Proof. The fact that the edges in $\{e^* \in E(G^*) : e \notin E(T)\}$ form a spanning tree of G^* is a well-known property of planar maps. It follows from two observations. Firstly, there are $|E(G)| - (|V(G)| - 1) = |F(G)| - 1 = |V(G^*)| - 1$ edges in this set. Secondly, the graph formed by these edges has no cycles (otherwise, this cycle separates a pair of faces, which means that the corresponding vertices of G are in different connected components of T; but T is connected, a contradiction).

We now show that T^* satisfies the condition of the lemma. First observe that going around T^* is equivalent to going around T in *clockwise* direction (Figure 6.6, left). This can be seen locally, by looking at each corner of T^* (Figure 6.6, right). This local picture also shows that crossing an edge from $E(G^*) \setminus E(T^*)$ at a black (white) corner of G^* corresponds to going along the left (right) side of the dual edge in T.

Going around T^* starting at a corner of the face of G^* with label i corresponds to going *clockwise* around T starting at a corner of the vertex with label i. Since all edges of T are oriented away from vertex i, when going clockwise around T each edge of T is first traversed on its left, then on its right. It means that when going around T^*, each edge in $E(G^*) \setminus E(T^*)$ is first crossed at a black corner, then at a white corner.

To prove that the correspondence $T \leftrightarrow T^*$ is bijective we provide the inverse map. Given T^*, the duals of edges in $E(G^*) \setminus E(T^*)$ form a spanning tree T of $\overrightarrow{G}$ by the same reasoning as before. By using the equivalence between going around T^* and T, the condition on T^* translates into the fact that all edges of T are oriented away from the vertex i. Finally, the fact that the two maps $T \to T^*$ and $T^* \to T$ are inverses of each other is straightforward. $\square$

6.3.2 Definition of $B^{d'}_{k,l}(L)$ and its volume

Let $k, l, n \geq 1$, $L \in \mathbb{R}^k_{>0}$ and let $d' = (d'_1, \ldots, d'_n)$ be a composition of $k + l + n - 1$ of length n.

Remark 6.11. *Note (!) that the weight of the composition d' is bigger by 1 than the weight of the composition d.*

Let $B^{d'}_{k,l}(L)$ denote the set of triples $(\overline{T}, f_\square, f_\blacksquare)$, where

- $\overline{T} \in \mathcal{RG}^*_{0,(k,n)}$ is a (non-rooted) vertex-bicolored plane tree with k black vertices labeled form 1 to k and n white vertices labeled from 1 to n; we will denote its vertices by their labels with a black or white box in the superscript to indicate color $(1^\blacksquare, 2^\blacksquare, \ldots, 1^\square, 2^\square, \ldots)$.
- $f_\blacksquare$ is a function from the set of black corners of $\overline{T}$ to $\mathbb{R}_{>0}$ such that for every $i = 1, \ldots, k$:
$$\sum_{c \text{ is a corner of } i^\blacksquare} f_\blacksquare(c) = L_i. \tag{6.2}$$
- $f_\square$ is a function which assigns to each white corner of $\overline{T}$ an ordered (possibly empty) sequence of distinct elements from $\{1, \ldots, l\}$ such that the values of $f_\square$ are pairwise disjoint, form a partition of $\{1, \ldots, l\}$, and such that for every $i = 1, \ldots, n$:
$$\sum_{c \text{ is a corner of } i^\square} (|f_\square(c)| + 1) = d'_i, \tag{6.3}$$
  where $|\cdot|$ denotes the length of a sequence.

By (6.2), the set $B^{d'}_{k,l}(L)$ can be identified with a disjoint union of products of interiors of scaled simplices:

$$B^{d'}_{k,l}(L) = \bigsqcup_{\overline{T}, f_\square} \prod_{i=1}^{k} \left(L_i \cdot \operatorname{int} \Delta_{\deg_{\overline{T}}(i^\blacksquare)} \right), \tag{6.4}$$

where the union is over all $\overline{T} \in \mathcal{RG}^*_{0,(k,n)}$ and all $f_\square$ satisfying (6.3), and where by Δ_N we denote the standard simplex in $\mathbb{R}^N$, i.e. the convex hull of the N standard unit vectors.

Each scaled simplex $L_i \cdot \operatorname{int} \Delta_{\deg_{\overline{T}}(i^{\blacksquare})}$ lies in the affine subspace of $\mathbb{R}^{\deg_{\overline{T}}(i^{\blacksquare})}$ defined by the equation $x_1 + \ldots + x_{\deg_{\overline{T}}(i^{\blacksquare})} = L_i$, where x_j are the standard coordinates. The tangent space to this subspace at any point is $x_1 + \ldots + x_{\deg_{\overline{T}}(i^{\blacksquare})} = 0$. Equip the subspace $x_1 + \ldots + x_{\deg_{\overline{T}}(i^{\blacksquare})} = L_i$ with the Lebesgue volume form normalized in such a way that the integer lattice of $x_1 + \ldots + x_{\deg_{\overline{T}}(i^{\blacksquare})} = 0$ has covolume 1. Then the volume of the scaled simplex $L_i \cdot \operatorname{int} \Delta_{\deg_{\overline{T}}(i^{\blacksquare})}$ with respect to this volume form is

$$\frac{L_i^{\deg_{\overline{T}}(i^{\blacksquare})-1}}{(\deg_{\overline{T}}(i^{\blacksquare})-1)!}.$$

Call the *volume of* $B^{d'}_{k,l}(L)$ the sum of the volumes of all the constituent products of scaled simplices. The important property of $B^{d'}_{k,l}(L)$ is that its volume is easy to compute.

Lemma 6.12. *For all k, l, n, d' and L as above, we have*

$$\operatorname{Vol} B^{d'}_{k,l}(L) = (k+l-1)! \cdot \left(\sum_{i=1}^{k} L_i\right)^{n-1},$$

independently of d'.

Proof. Let $s^{\blacksquare}_1, \ldots, s^{\blacksquare}_k$ be the degrees of black vertices of $\overline{T}$ and $s^{\square}_1, \ldots, s^{\square}_n$ be the degrees of its white vertices. Then clearly $s^{\blacksquare}_1 + \ldots + s^{\blacksquare}_k = s^{\square}_1 + \ldots + s^{\square}_n = k + n - 1$. For $j = 1, \ldots, k+n-1$, let $a^{\blacksquare}_j$ be the number of $s^{\blacksquare}_i$ equal to j and let $a^{\square}_j$ be the number of $s^{\square}_i$ equal to j. It is known ([GJ92]) that the number of *rooted unlabeled* vertex bicolored plane trees with $a^{\blacksquare}_j$ black vertices of degree j and $a^{\square}_j$ white vertices of degree j is equal to

$$(k+n-1) \cdot \frac{(k-1)!}{a^{\blacksquare}_1! \cdots a^{\blacksquare}_{k+n-1}!} \cdot \frac{(n-1)!}{a^{\square}_1! \cdots a^{\square}_{k+n-1}!}.$$

Labelling the vertices in all possible $a^{\blacksquare}_1! \cdots a^{\blacksquare}_{k+n-1}! \cdot a^{\square}_1! \cdots a^{\square}_{k+n-1}!$ ways and forgetting the rooting, we obtain that the number of ways to choose T with given vertex degrees is simply $(k-1)!(n-1)!$, independently of $s^{\blacksquare}_i$ and $s^{\square}_i$.

To choose $f_\square$, we first choose the lengths of the sequences $f_\square(c)$. By (6.3), the number of choices is equal to the product over $i = 1, \ldots, n$ of the number of compositions of d_i of length $s^{\square}_i$. This last number is the binomial coefficient $\binom{d_i-1}{s^{\square}_i-1}$. When the lengths of $f_\square(c)$ are fixed, there are clearly $l!$ ways to distribute the labels in $\{1, \ldots, l\}$ among the sequences $f_\square(c)$.

The volume of the product of scaled simplices in (6.4) corresponding to the choice of $f_{\blacksquare}$ is equal to

$$\prod_{i=1}^{k} \frac{L_i^{s_i^{\blacksquare}-1}}{(s_i^{\blacksquare}-1)!}.$$

Putting all these observations together, we get

$$\begin{aligned}
\operatorname{Vol} B_{k,l}^{d'}(L) \\
&= (k-1)!(n-1)!l! \cdot \sum_{s_i^{\blacksquare}, s_i^{\square}} \prod_{i=1}^{n} \binom{d_i-1}{s_i^{\square}-1} \cdot \prod_{i=1}^{k} \frac{L_i^{s_i^{\blacksquare}-1}}{(s_i^{\blacksquare}-1)!} \\
&= (k-1)!l! \cdot \left(\sum_{s_i^{\square}} \prod_{i=1}^{n} \binom{d_i-1}{s_i^{\square}-1} \right) \cdot \left(\sum_{s_i^{\blacksquare}} \frac{(n-1)!}{\prod_{i=1}^{k}(s_i^{\blacksquare}-1)!} \prod_{i=1}^{k} L_i^{s_i^{\blacksquare}-1} \right) \\
&= (k-1)!l! \cdot \binom{k+l-1}{k-1} \cdot \left(\sum_{i=1}^{k} L_i \right)^{n-1} = (k+l-1)! \cdot \left(\sum_{i=1}^{k} L_i \right)^{n-1},
\end{aligned}$$

where, in the second to last equality, we have used the multinomial expansion (for the second sum) and the following combinatorial identity (for the first sum):

$$\sum_{\substack{x_1+\ldots+x_N=S,\\ x_i \geq 0}} \prod_{i=1}^{N} \binom{y_i}{x_i} = \binom{y_1+\ldots+y_N}{S}.$$

This identity follows from the following trivial observation: choosing S elements out of $y_1+\ldots+y_N$ is equivalent to choosing, for each $i=1,\ldots,N$, x_i elements out of y_i, for some composition $(x_1,\ldots,x_N)$ of S. □

6.3.3 Definition of ψ

We now define the map

$$\psi : A_{k,l,i_{root}}^{(d_1,\ldots,d_n)}(L,L') \to B_{k,l}^{(d_1,\ldots,d_{i_{root}}+1,\ldots,d_n)}(L).$$

This will be done in two steps. Let $(G,T,w) \in A_{k,l,i_{root}}^{(d_1,\ldots,d_n)}(L,L')$.

Step 1: applying Bernardi bijection

Apply the Bernardi bijection (described in section 6.2.3) to the pair (G,T) (forgetting the labels of the vertices and faces for a moment). Let (T',T'') be the resulting pair of unlabeled plane rooted trees. Note that T'' is vertex-bicolored by definition.

Lemma 6.13. *In the process of "exploding" G to produce T', each white vertex of G is not modified (i.e. not broken into several vertices of T'). In particular, the black vertices of T'' corresponding to the white vertices of G have degree 1.*

Proof. By the definition of the explosion of a vertex, the vertex is not modified if and only if its indegree in the corresponding orientation of G is one. We thus have to prove the the indegree of each white vertex is one.

Recall the definition of the orientation of G corresponding to the spanning tree T. Combining this with the property of the spanning tree T, we see that in this orientation each edge in $E(G) \setminus E(T)$ is oriented from its white to its black extremity (see Figure 6.7, first row, left). It means that these edges do not contribute to the indegrees of white vertices. The edges in $E(T)$ are oriented away from the root vertex, hence the indegree of each white vertex is indeed one.

The last claim follows directly from the construction of T''. $\square$

Color the vertices of T' into black and white, according to whether they are a result of the explosion of a black or a white vertex of G (this makes T' vertex-bicolored). Label its white vertices by the label of the corresponding vertex of G (by Lemma 6.13 white vertices of G are not modified during explosion), see Figure 6.7, second row.

Step 2: encoding all data into a single decorated tree

We now construct the triple $(\overline{T}, f_\square, f_\blacksquare) = \psi(G, T, w)$.

$\overline{T} \in \mathcal{RG}^*_{0,(k,n)}$ is constructed from T'' by forgetting the rooting, removing the (black) leaves corresponding to white vertices of G, and labelling the rest of its vertices by the labels of the corresponding black vertices or faces of G (so these labels are $1^\blacksquare, \dots, k^\blacksquare$ and $1^\square, \dots, n^\square$), see Figure 6.7, third row.

Let $\overline{c}_1^\blacksquare, \overline{c}_1^\square, \dots, \overline{c}_{k+n-1}^\blacksquare, \overline{c}_{k+n-1}^\square$ be the corners of $\overline{T}$ written in (some) *clockwise* order, with superscript indicating the color. Let also $c'_1, \dots, c'_{k+n-1}$ be the corners of first visit to the black vertices of T' corresponding to the corners $\overline{c}_1^\blacksquare, \dots, \overline{c}_{k+n-1}^\blacksquare$ of $\overline{T}$ (they are in counterclockwise order around T').

There is an obvious identification between the edges of G and T'. In particular, the metric w on G gives rise to a metric w' on T'. $f_\blacksquare$ is then defined as follows: for each $i = 1, \dots, k+n-1$, $f_\blacksquare(\overline{c}_i^\blacksquare)$ is equal to the vertex perimeter (with respect to w') of the black vertex of T' incident to the corner c'_i.

$f_\square$ is defined as follows: for each $i = 1, \dots, k+n-1$, $f_\square(\overline{c}_i^\square)$ is equal to the sequence of labels of the corners of *last* visit to *white* vertices of T' (when

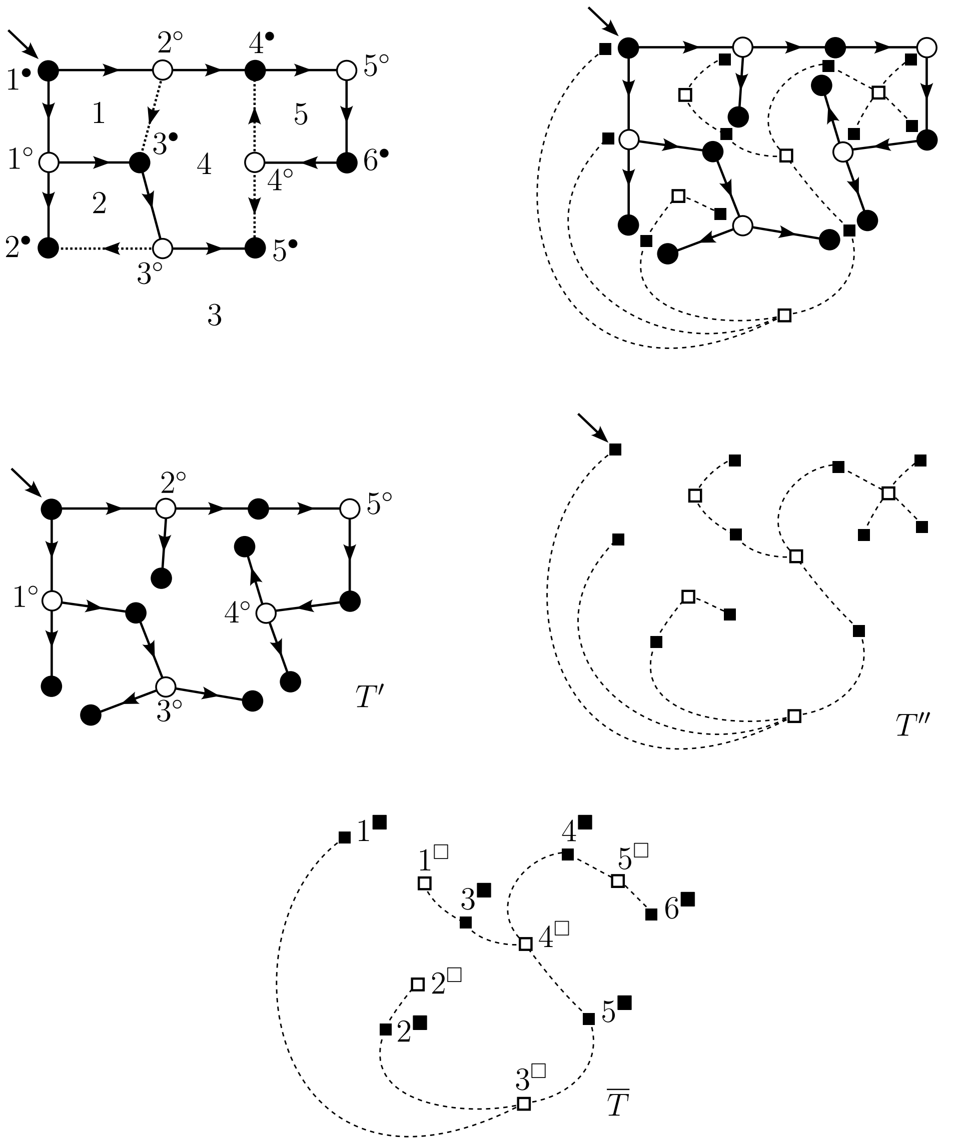


Figure 6.7: Steps of the map ψ. *First row, left*: a rooted labeled vertex-bicolored map with orientation corresponding to some spanning tree (dotted edges do not belong to the tree). *First row, right*: applying Bernardi bijection. *Second row*: resulting trees T' and T''. *Third row*: the tree $\overline{T}$ constructed from T''.

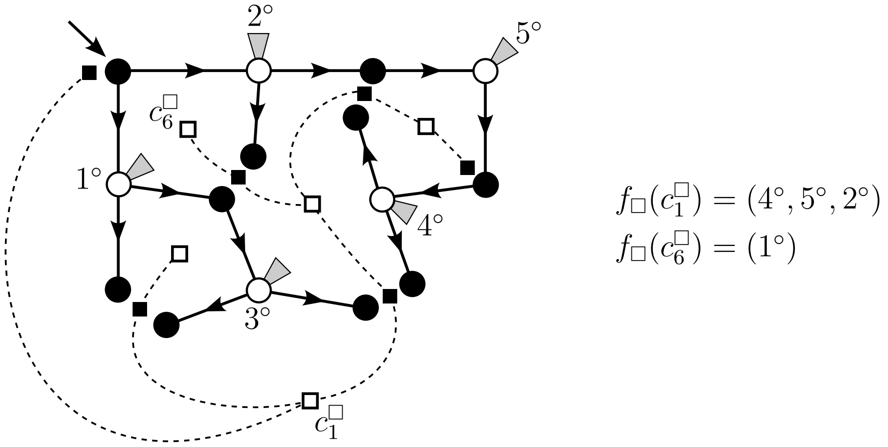


Figure 6.8: Definition of $f_{\square}$ (decoration of the white corners of $\overline{T}$).

going counterclockwise around T') between c'_i and c'_{i+1} (indices are modulo $k+n-1$). See Figure 6.8.

Lemma 6.14. *The map ψ is well-defined, i.e. $f_{\blacksquare}$ and $f_{\square}$ constructed above satisfy the conditions (6.2) and (6.3), respectively, with*

$$d' = (d_1, \ldots, d_{i_{root}} + 1, \ldots, d_n).$$

Proof. The condition (6.2) is satisfied by $f_{\blacksquare}$, because the black vertices of T' whose first visit corners correspond to the corners of the vertex $i^{\blacksquare}$ of $\overline{T}$, come from exploding the vertex $i^{\bullet}$ of G. So the sum of their vertex perimeters is the vertex perimeter of $i^{\bullet}$, which is L_i.

We now prove that $f_{\square}$ satisfies the condition (6.3) with $d' = (d_1, \ldots, d_{i_{root}} + 1, \ldots, d_n)$. For this, recall the definition of the prefix-postfix marking of a tree (Definition 5.1). Consider the prefix-postfix marking of T'. By construction, the markers of black vertices are in the corners $c'_1, \ldots, c'_{k+n-1}$ and there are $|f_{\square}(\overline{c}_i^{\square})|$ white markers between c'_i and c'_{i+1}. Suppose c'_{i+1} is not the marker of the root vertex. Then by Lemma 5.11, the distance (counterclockwise along the boundary of T') between c'_i and c'_{i+1} is equal to $1 + 2(|f_{\square}(\overline{c}_i^{\square})| - 1) + 3 = 2(|f_{\square}(\overline{c}_i^{\square})| + 1)$. If c'_{i+1} is the marker of the root vertex, then this distance is smaller by 2: it is equal to $1 + 2(|f_{\square}(\overline{c}_i^{\square})| - 1) + 1 = 2(|f_{\square}(\overline{c}_i^{\square})|)$, because the distance between the last white marker of T' and the black marker of the root vertex of T' is always 1 (the root edge connects them).

Finally, if the corners around the vertex $i^{\square}$ of $\overline{T}$ are $\overline{c}_{j_1}^{\square}, \ldots, \overline{c}_{j_N}^{\square}$, then the parts of the boundary of T' between pairs of corners (c'_{j_1}, c'_{j_1+1}), (c'_{j_2}, c'_{j_2+1}), $\ldots$, (c'_{j_N}, c'_{j_N+1}), will together form the boundary of G with label i when T' is reglued back to form G. Since the degree of the boundary of G with label

i is $2d_i$, we obtain for $i \neq i_{root}$

$$\sum_{c \text{ is a corner of } i^{\square}} 2(|f_{\square}(c)| + 1) = 2d_i = 2d'_i,$$

and for $i = i_{root}$

$$\sum_{c \text{ is a corner of } i^{\square}_{root}} 2(|f_{\square}(c)| + 1) = 2d_{i_{root}} + 2 = 2d'_{i_{root}},$$

which is equivalent to (6.3). □

Lemma 6.15. *The map* $\psi : A^{(d_1,\dots,d_n)}_{k,l,i_{root}}(L, L') \to B^{(d_1,\dots,d_{i_{root}}+1,\dots,d_n)}_{k,l}(L)$ *is affine and volume-preserving.*

Proof. We first prove that ψ is affine. By construction, when G and T are fixed, the tree $\overline{T}$ and the function $f_{\square}$ are also fixed. Hence the restriction of ψ to (one of the copies of) $M_G(L, L')$ maps to a fixed product of simplices in $B^{(d_1,\dots,d_{i_{root}}+1,\dots,d_n)}_{k,l}(L)$. The values of $f_{\blacksquare}$ are vertex perimeters of T' for the metric w', and so they are sums of certain lengths of edges of G with respect to the metric w. These lengths, in turn, are linear functions on the polytopes $M_G(L, L')$. Thus the restriction of ψ to $M_G(L, L')$ is affine.

We now show that ψ is volume-preserving. Recall from section 2.2.4 that $M_G(L, L')$ lies in the affine subspace $\mathrm{vp}_G^{-1}(L, L')$ of $\mathbb{R}^{E(G)}$. The tangent space to $\mathrm{vp}_G^{-1}(L, L')$ at any point is $\mathrm{vp}_G^{-1}(0, 0)$. $\mathrm{vp}_G^{-1}(L, L')$ is equipped with the Lebesgue volume form normalized in such a way that the covolume of the integer lattice of $\mathrm{vp}_G^{-1}(0, 0)$ is 1. Recall also that the Lebesgue volume form on the products of simplices in $B^{(d_1,\dots,d_{i_{root}}+1,\dots,d_n)}_{k,l}(L)$ is also normalized so that the covolume of the integer lattice in any tangent space is equal to 1.

It is thus enough to show that the two polytopes have the same dimension, and that the differential of ψ identifies integer vectors of the corresponding tangent spaces.

The dimension of $M_G(L, L')$ is $\dim \mathrm{vp}_G^{-1}(L, L') = |E(G)| - |V(G)| + 1 = (k + l + n - 2) - (k + l) + 1 = n - 1$, by Lemma 2.6. The dimension of the product of simplices is $\sum_{i=1}^{k} (\deg_{\overline{T}}(i^{\blacksquare}) - 1) = (k + n - 1) - k = n - 1$ as well.

The tangent vectors dw to $M_G(L, L')$ are metrics on G with vertex perimeters $(0, 0)$. If dw is integer, then so is the metric dw' on T' and its vertex perimeters $d\psi(dw)$; conversely, if the vertex perimeters $d\psi(dw)$ of dw' are integer, then the metric dw' itself is integer (by Lemma 2.5), and so is dw. □

6.3.4 Collecting different maps ψ

Fix a composition $d' = (d'_1, \dots, d'_n)$ of $k + l + n - 1$ of length n, with all $d'_i \geq 2$. Observe that there are several different values of (d, i_{root}) such that the corresponding map ψ maps from $A^{d}_{k,l,i_{root}}(L,L')$ to $B^{d'}_{k,l}(L)$. Namely, these are the pairs

$$d = (d'_1, \dots, d'_i - 1, \dots, d'_n),\ i_{root} = i,\ i = 1, \dots, n.$$

We collect these maps into a single map (which we also denote as ψ by slight abuse of notation):

$$\psi : \bigsqcup_{i=1}^{n} A_{k,l,i}^{(d'_1,\dots,d'_i-1,\dots,d'_n)}(L,L') \to B^{d'}_{k,l}(L). \tag{6.5}$$

The following is the central claim in the proof of Theorem 6.5.

Proposition 6.16. *If $(L,L') \in H^{\circ}_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$, then ψ is a bijection onto a subset of $B^{d'}_{k,l}(L)$ of full measure.*

Proof. We first prove that ψ is injective. Suppose that $\psi(G,T,w) = (\overline{T}, f_{\square}, f_{\blacksquare})$. We show how to reconstruct G, T, w from $(\overline{T}, f_{\square}, f_{\blacksquare})$.

Let T', T'' be the trees from the Step 1 of the construction of $\psi(G,T,w)$ and let w' be the metric on T' induced from w. Let also $\overline{c}^{\blacksquare}_1, \dots, \overline{c}^{\blacksquare}_{k+n-1}$ be the corners of $\overline{T}$ written in (some) *clockwise* order and let $c'_1, \dots, c'_{k+n-1}$ be the corresponding corners of first visit to the black vertices of T'.

Recall the definitions of the prefix-postfix sequence of a tree (Definition 5.2) and of its cyclic equivalence class (denoted by square brackets). If we label the black vertices of T' incident to the corners $c'_1, \dots, c'_{k+n-1}$ as $1^{\bullet}, \dots, (k+n-1)^{\bullet}$ respectively, then, by construction, the prefix-postfix sequence of T' satisfies

$$[\pi(T')] = \left[\left(1^{\bullet}, f_{\square}(\overline{c}^{\square}_1), 2^{\bullet}, f_{\square}(\overline{c}^{\square}_2), \dots, (k+n-1)^{\bullet}, f_{\square}(\overline{c}^{\square}_{k+n-1})\right)\right], \tag{6.6}$$

and T' is positive at the point

$$\left(f_{\blacksquare}(\overline{c}^{\blacksquare}_1), \dots, f_{\blacksquare}(\overline{c}^{\blacksquare}_{k+n-1}); L'_1, \dots, L'_l\right) \tag{6.7}$$

(these are the vertex perimeters of T' with respect to the metric w'). Note that these two data are uniquely determined by the triple $(\overline{T}, f_{\square}, f_{\blacksquare})$. Moreover, if follows from Theorem 5.3 that T' is the unique rooted plane tree satisfying these two properties (if there were two trees satisfying (6.6) positive at the point (6.7), they would be both positive on some neighborhood of this point, hence also at some point of $H^{\circ}_{k+n-1,l} \cap (\mathbb{R}^{k+n-1}_{>0} \times \mathbb{R}^l_{>0})$, which is impossible by Theorem 5.3). It means that T' (with this labeling of black vertices) *is uniquely determined by* $(\overline{T}, f_{\square}, f_{\blacksquare})$. It is then enough to forget the labels of the black vertices to recover T' itself.

The metric w' is the *unique* metric on T' with vertex perimeters (6.7), by Lemma 2.5.

We now show that T'' is also uniquely determined. Indeed, to reconstruct T'' perform the following operations (read Figure 6.7 in reverse):

- forget the labels of vertices of $\overline{T}$
- for each $i = 1, \dots, k+n-1$, let s_i be the number of corners of *first* visit to *white* vertices of T' between the corners c'_i and c'_{i+1}; then, in the white corner $\overline{c}_i^{\square}$ of $\overline{T}$, glue s black leaves;
- root $\overline{T}$ at the black corner $\overline{c}_i^{\blacksquare}$ such that c'_i is the root corner of T'.

Now, the *unlabeled* version of G, T is uniquely determined from T', T'' by the inverse of the Bernardi bijection (regluing). The labels of white vertices of G are reconstructed from the labels of white vertices of T'. The labels of the black vertices and the faces of G are reconstructed from the labels of the corresponding vertices of $\overline{T}$. Finally, the metric w on G is uniquely determined by the metric w' on T'. Thus ψ is indeed injective.

We now prove that for almost all $(\overline{T}, f_\square, f_\blacksquare) \in B^{d'}_{k,l}(L)$ there is a triple (G, T, w) such that $\psi(G, T, w) = (\overline{T}, f_\square, f_\blacksquare)$.

We say that $(\overline{T}, f_\square, f_\blacksquare) \in B^{d'}_{k,l}(L)$ is *generic* if the vector (6.7) lies in $H^\circ_{k+n-1,l} \cap (\mathbb{R}^{k+n-1}_{>0} \times \mathbb{R}^l_{>0})$. We claim that almost all triples $(\overline{T}, f_\square, f_\blacksquare)$ are generic. Indeed, fix $\overline{T}, f_\square$ and consider a relation of the form $\sum_{i \in I} f_\blacksquare(\overline{c}_i^{\blacksquare}) = \sum_{j \in J} L'_j$ for some I, J. If for some $i_0 \in \{1, \dots, k\}$ there are two corners $\overline{c}_{i_1}^{\blacksquare}, \overline{c}_{i_2}^{\blacksquare}$ around the vertex $i_0^{\blacksquare}$ of $\overline{T}$ with $i_1 \in I$ and $i_2 \notin I$, then one could slightly modify $f_\blacksquare$ to break this relation. Hence the set of $f_\blacksquare$ satisfying this relation lies in a transverse intersection with a hyperplane, and so is of measure zero. If there are no such i_0, i_1, i_2 as above, then we must have $\sum_{i \in I} f_\blacksquare(\overline{c}_i^{\blacksquare}) = \sum_{i_0 \in I_0} L_{i_0} = \sum_{j \in J} L'_j$ for some I_0. But this is impossible since $(L, L') \in H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$ by assumption.

Let now $(\overline{T}, f_\square, f_\blacksquare)$ be a generic triple. We will construct G, T, w such that $\psi(G, T, w) = (\overline{T}, f_\square, f_\blacksquare)$. By Theorem 5.3 there is a unique rooted plane tree T' satisfying (6.6) and positive at the point (6.7). Let w' be the unique metric on T' with vertex perimeters (6.7). Forget the labels of black vertices of T'. Let also T'' be the tree constructed in the same way as in the proof of injectivity above.

One can apply the inverse of the Bernardi bijection (regluing) to T', T'' to get an unlabeled pair G, T. By the construction of T', T'', during this regluing each white vertex of T' is glued only once to some leaf of T'', in particular it is not glued to any other vertices of T'. It means that the vertex-bicoloring of T'

induces a vertex-bicoloring of G. It also means that the white-vertex labeling of T' induces a white-vertex labeling of G. The labels of the black vertices and of the faces of G are induced by the labeling of the corresponding vertices of $\overline{T}$. Because $f_\square$ satisfies (6.3), the degrees of faces of G are $2d'_1, \ldots, 2(d'_{i_{root}} - 1), \ldots, 2d'_n$, for some $i_{root} \in \{1, \ldots, n\}$, by the same reasoning as in the proof of Lemma 6.14. Thus $G \in \mathcal{RG}_{0,(k,l)}^{(2d'_1,\ldots,2(d'_{i_{root}}-1),\ldots,2d'_n),*,root}$.

The metric w' on T induces a metric w on G. Because $f_\blacksquare$ satisfies (6.2), the vertex perimeters of w are $(L_1, \ldots, L_n; L'_1, \ldots, L'_l)$.

Finally, we have to check that the spanning tree T of G satisfies the condition of Definition 6.9 with i being the label of the root face. Recall that the edges of T' are oriented away from its root. By using again the fact that during regluing of T' and T'', the white vertices of T' are not glued to any other vertices of T', we see that in the induced orientation of G, all the white vertices have indegree 1. By looking at the construction of this orientation from (G, T), we see that all the edges in $E(G) \setminus E(T)$ must be oriented towards their black extremities. Hence, when going around T starting from the root of G, we first cross these edges at their black extremities, as desired.

Thus we have constructed $(G, T, w) \in A_{k,l,i_{root}}^{(d'_1,\ldots,d'_{i_{root}}-1,\ldots,d'_n)}(L, L')$ such that $\psi(G, T, w) = (\overline{T}, f_\square, f_\blacksquare)$. □

Corollary 6.17. *Let $k, l, n \geq 1$, let $d' = (d'_1, \ldots, d'_n)$ be a composition of $k+l+n-1$ of length n, with all $d'_i \geq 2$. Let C be a connected component of $H^\circ_{k,l} \cap (\mathbb{R}^k_{>0} \times \mathbb{R}^l_{>0})$ and let $(L, L') \in C$. Then*

$$\sum_{i=1}^{n} (d'_i - 1) \cdot top_C \left(\widetilde{\mathcal{P}}_{k,l}^{0,(2d'_1,\ldots,2d'_i-2,\ldots,2d'_n)} \right) (L, L') = (k+l-1)! \cdot \left(\sum_{i=1}^{k} L_i \right)^{n-1}. \tag{6.8}$$

Proof. ψ is volume-preserving by Lemma 6.15. Hence, by Proposition 6.16, the volumes of the domain and the codomain of ψ in (6.5) are equal. The volume of each component of the domain is given by (6.1). The volume of the codomain is given by Lemma 6.12. □

We are now in position to prove Theorem 6.5.

Proof of Theorem 6.5. Fix k, l, n, C and (L, L'). We prove the Theorem for all compositions $d = (d_1, \ldots, d_n)$ of $k+l+n-2$ of length n. The proof is by (descending) induction on the maximal element M of d.

The base case is $M = M_0 = k+l-1$, which is attained on the composition $d_0 = (M_0, 1, 1, \ldots, 1)$ and all of its permutations. The statement is true in this base case by Corollary 6.7 applied with $g = 0$ and using the fact that $P^0_{H_{k,l}}(L, L') = (k + l - 2)!$ (Proposition 3.3).

For the step of induction, assume that the Theorem is true for all d with $\max d = M \geq 2$. Let d be such that $\max d = M - 1$. Without loss of generality, assume that $d_1 = M - 1$. Apply Corollary 6.17 to $d' = (d_1 + 1, d_2, \ldots, d_n)$. The first term in the left-hand side of (6.8) becomes $(M - 1) \cdot top_C \left(\widetilde{\mathcal{P}}^{0,2d}_{k,l}\right)(L, L')$. All the other terms are of the form $(d_i - 1) \cdot (k + l - 2)! \cdot \left(\sum_{i=1}^{k} L_i\right)^{n-1}$ by induction hypothesis (the maximal element of the corresponding compositions is M). Moving these terms to the right-hand side of (6.8) we get

$$(M - 1) \cdot top_C \left(\widetilde{\mathcal{P}}^{0,2d}_{k,l}\right)(L, L')$$
$$= (k + l - 1 - \sum_{i=2}^{n}(d_i - 1)) \cdot (k + l - 2)! \cdot \left(\sum_{i=1}^{k} L_i\right)^{n-1}$$
$$= (M - 1) \cdot (k + l - 2)! \cdot \left(\sum_{i=1}^{k} L_i\right)^{n-1}.$$

□

Chapter 7

Metric ribbon graphs with odd vertex degrees

This chapter presents my contribution to a joint project with Eduard Duryev and Élise Goujard (paper in preparation). The aim of this project is to generalize the combinatorial formula of [DGZZ21] for the volumes of principal strata of quadratic differentials (described in section 1.3.4) to the strata with the degrees of all singularities being odd.

Ultimately, this boils down to the study of the functions $\mathcal{N}^{\underline{k}}_{g,n}(L)$, which count (non-face-bicolored) metric ribbon graphs with given odd degrees of vertices and with fixed perimeters of boundary components.

These counting functions satisfy an analog of Theorem 1.8 (also due to Kontsevich): their top-degree terms (outside of the walls) are in fact polynomial and the coefficients of these polynomials are certain intersection numbers (Theorem 7.2). However, the behavior of $\mathcal{N}^{\underline{k}}_{g,n}(L)$ on the walls and their intersections has not been studied. Yet, our application requires the knowledge of $\mathcal{N}^{\underline{k}}_{g,n}(L)$ on these subspaces.

The main result of this chapter is Theorem 7.3, which states that the top-degree terms of $\mathcal{N}^{\underline{k}}_{g,n}$ on a certain family of walls and their intersections are also polynomials and can be recursively computed, thus strengthening Theorem 1.8. The proof relies on the accurate study of degenerations of ribbon graphs in $\mathcal{RG}^{\underline{k},*}_{g,n}$, and eventually boils down to the proof of Proposition 7.6 (which is my contribution), giving a formula for the coefficients counting certain particular degenerations, first found experimentally by Eduard and Élise. This latter proof is combinatorial and uses a Prüfer-code-style bijection for degenerated ribbon graphs.

We start by stating the main theorem in section 7.1.1. Then, in section 7.1.2, we discuss degenerations of ribbon graphs in $\mathcal{RG}^{\underline{k},*}_{g,n}$ and explain how the main theorem follows from Proposition 7.6. Section 7.1.3 explains the application of these results to the joint project with Eduard and Élise. The proofs follow in section 7.2.

7.1 Introduction

7.1.1 Main result

In this chapter we call a partition *odd* if all of its parts are odd.

Let $g \geq 0$, $n \geq 1$ and let $\underline{k} = [k_1, \ldots, k_s]$ be an odd partition. Denote by m_i the number of parts in $\underline{k}$ equal to $2i+1$, i.e. $\underline{k} = [1^{m_0}, 3^{m_1}, \ldots]$. Recall that $\mathcal{RG}_{g,n}^{\underline{k}}$ denotes the set of isomorphism classes of genus g ribbon graph with n boundary components labeled from 1 to n, and s *unlabeled* vertices of degrees $k_1, \ldots, k_s$. The dual family $\mathcal{RG}_{g,n}^{\underline{k},*}$ consists of genus g ribbon graph with n vertices labeled from 1 to n, and s *unlabeled* faces of degrees $k_1, \ldots, k_s$.

Recall also the duality Convention 1.7. By this convention, the counting function $\mathcal{N}_{g,n}^{\underline{k}}(L)$ of the family $\mathcal{RG}_{g,n}^{\underline{k}}$ can be written in terms of dual graphs as

$$\mathcal{N}_{g,n}^{\underline{k}}(L) = \sum_{G \in \mathcal{RG}_{g,n}^{\underline{k},*}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{N}_G(L),$$

where $\mathcal{N}_G(L)$ denotes here the number of integer metrics on G with *vertex* perimeters given by L. From now on we will use this alternative definition.

By Euler's formula, $\mathcal{RG}_{g,n}^{\underline{k},*}$ is non-empty if and only if

$$n - \frac{1}{2} \cdot \left(\sum_{i=1}^{s} k_i \right) + s = 2 - 2g \iff \frac{1}{2} \sum_{i=0}^{\infty} m_i (2i-1) = 2g - 2 + n. \quad (7.1)$$

Denote

$$M = \sum_{i=0}^{\infty} m_i (i-1). \quad (7.2)$$

Recall the definitions of and the notations for the polyhedral subdivision $\mathcal{PS}_n$ of $\mathbb{R}^n$, the set of walls $\mathcal{W}_n$ and the set of their intersections $\overline{\mathcal{W}_n}$, introduced in section 2.2.1.

First of all, the general nature of the counting functions $\mathcal{N}_{g,n}^{\underline{k}}$ is given by

Proposition 7.1. *Fix $g \geq 0$, $n \geq 1$, an odd partition $\underline{k}$. Then for every subspace $W \in \overline{\mathcal{W}_n}$ and every connected component C of $\mathbb{R}^n_{>0} \cap W^\circ$ the function $\mathcal{N}_{g,n}^{\underline{k}}(L)$ is either identically zero or is a polynomial of degree $6g-6+2n-2M$ for $L \in C$ and in a fixed coset of $2\mathbb{Z}^n \subset \mathbb{Z}^n$.*

Proof. Any connected component C of $\mathbb{R}^n_{>0} \cap W^\circ$ is an open cell of the polyhedral subdivision $\mathcal{PS}_n$ of $\mathbb{R}^n$. Thus, by Proposition 2.17, the contribution of any $G \in \mathcal{RG}^{\underline{k},*}_{g,n}$ to $\mathcal{N}^{\underline{k}}_{g,n}(L)$ for $L \in C$ and in a fixed coset of $2\mathbb{Z}^n \subset \mathbb{Z}^n$ is either zero or a polynomial of degree $|E(G)| - |V(G)| = \frac{1}{2}\sum_{i=1}^{\infty} m_i(2i+1) - n$. From (7.1) we have $\frac{3}{2}\sum_{i=0}^{\infty} m_i(2i-1) = 6g-6+3n$, and so $\frac{1}{2}\sum_{i=1}^{\infty} m_i(2i+1) - n = 6g-6+2n-2M$. $\square$

The reason why we write the degree of $\mathcal{N}^{\underline{k}}_{g,n}$ as $6g-6+2n-2M$ is because $2M$ is the codimension of the locus of metric ribbon graphs whose underlying ribbon graph is in $\mathcal{RG}^{\underline{k}}_{g,n}$, in the ambient combinatorial moduli space $\mathcal{M}^{comb}_{g,n}$ of dimension $6g-6+2n$. See section 7.1.4 for more details.

We now pass to the study of the top-degree terms of $\mathcal{N}^{\underline{k}}_{g,n}$.

As for the case of the counting functions for trivalent ribbon graphs $\mathcal{N}^{[3^{4g+2n-4}]}_{g,n}$ (see section 1.2.4), Kontsevich also proves in [Kon92] that the top-degree terms of the functions $\mathcal{N}^{\underline{k}}_{g,n}$ are polynomial. However, for $\underline{k}$ with at least one part larger than 3, this expression for the top-degree term is only valid outside of the walls in $\mathcal{W}_n$ (the reason for this is explained in the next section, see Remark 7.5).

Theorem 7.2 ([Kon92], section 3.3)**.** *Fix $g \geq 0$, $n \geq 1$ and an odd partition $\underline{k}$ satisfying (7.1). Let $\underline{k} = [1^{m_0}, 3^{m_1}, \dots]$ and $M = \sum_{i=0}^{\infty} m_i(i-1)$.*

There exists a homogeneous polynomial $N^{\underline{k}}_{g,n}$ of degree $6g-6+2n-2M$ such that for all $L \in \mathbb{Z}^n_{>0}$ outside of the hyperplanes in $\mathcal{W}_n$ and such that $\sum_{i=1}^{n} L_i$ is even, we have

$$\mathcal{N}^{\underline{k}}_{g,n}(L) = N^{\underline{k}}_{g,n}(L) + \textit{terms of lower degree.}$$

By Proposition 7.1, the "terms of lower degree" are given by a piecewise quasi-polynomial of degree less then $6g-6+2n-2M$.

In other terms, for every connected component C of $(\mathbb{R}^n)^\circ \cap \mathbb{R}^n_{>0}$ we have, independently of C,

$$top_C(\mathcal{N}^{\underline{k}}_{g,n})(L) = N^{\underline{k}}_{g,n}(L).$$

We call the polynomials $N^{\underline{k}}_{g,n}$ the *generalized Kontsevich polynomials.* These are polynomials of L_i^2 and their coefficients are certain intersection numbers. The precise definition is postponed to section 7.1.4.

Our main result is a generalization of Theorem 7.2 to a certain family of subspaces $W \in \overline{\mathcal{W}_n}$ that we now define.

Let $r \geq 1$ and let $\Pi = (I_0, I_1, J_1, \dots, I_r, J_r)$ be a sequence of $2r+1$ nonempty sets forming a partition of the set $\{1, \dots, n\}$. Let $W_\Pi \in \overline{\mathcal{W}_n}$ be the subspace defined by the equations

$$\sum_{i \in I_p} L_i = \sum_{j \in J_p} L_j, \;\; p = 1, \dots, r. \tag{7.3}$$

Theorem 7.3. *Fix $g \geq 0$, $n \geq 1$, and $\underline{k}$ an odd partition satisfying (7.1). Let $\underline{k} = [1^{m_0}, 3^{m_1}, \dots]$ and $M = \sum_{i=0}^{\infty} m_i(i-1)$. Let also Π be a partition of $\{1,\dots,n\}$ into $2r+1$ subsets, as above.*

Then there exists a homogeneous polynomial N_{W_Π} of degree $6g-6+2n-2M$ such that for all $L \in W_\Pi^\circ \cap \mathbb{Z}_{>0}^n$ and such that $\sum_{i=1}^n L_i$ is even, we have

$$\mathcal{N}_{g,n}^{\underline{k}}(L) = N_{W_\Pi}(L) + \textit{terms of lower degree}.$$

Moreover, the polynomials N_{W_Π} can be computed recursively.

We do not give here the explicit recursion for the polynomials N_{W_Π}. Nevertheless, the explanation of how to obtain it is given in the following section.

The idea of proof of Theorem 7.3 is to implement the general strategy for the computation of top-degree terms of the counting functions on the cells of $\mathcal{PS}_n$ of positive codimension, described in section 2.2.5. In the next section we explain the details of this implementation, and reduce the proof of Theorem 7.3 to a proof of Proposition 7.6, which gives the count of certain degenerations of ribbon graphs in $\mathcal{RG}_{g,n}^{\underline{k},*}$.

7.1.2 Degenerations of graphs in $\mathcal{RG}_{g,n}^{\underline{k},*}$

Choose any connected component C of $W_\Pi^\circ \cap \mathbb{R}_{>0}^n$ and an adjacent highest dimensional cell C_0 of $\mathcal{PS}_n$. The idea of proof of Theorem 7.3 is to compute explicitly the difference between $top_{C_0}\left(\mathcal{N}_{g,n}^{\underline{k}}\right)$ (the generalized Kontsevich polynomial $N_{g,n}^{\underline{k}}$) and $top_C\left(\mathcal{N}_{g,n}^{\underline{k}}\right)$ using the strategy of section 2.2.5, and to show that this difference is independent of C.

To this end, consider a ribbon graph $G \in \mathcal{RG}_{g,n}^{\underline{k},*}$ that degenerates when passing from C_0 to a point $L \in C$. It means that there is at least one static edge $e \in S(G)$ whose length becomes zero on C.

First note that all of these static edges should be bridges of G. Indeed, if one of these edges e were not a bridge, by Lemma 2.10 its length would be given by a linear function of the form $\frac{1}{2}(\sum_{i\in I} L_i - \sum_{i\in I^c} L_i)$ for some $I \subset \{1,\dots,n\}$. Thus we must have $\sum_{i\in I} L_i = \sum_{i\in I^c} L_i$ on C, and so on W_Π. However, this equation is not a linear combination of the defining equations (7.3) of W_Π (essentially because I_0 is non-empty), a contradiction.

Hence, if G degenerates into a disconnected graph $G_0 \sqcup G_1 \sqcup \ldots \sqcup G_m$, then G consists of the G_i connected together into a tree-like structure with the static edges of G whose lengths become zero on C. Such degenerations have the following properties (the proof is given in section 7.2.1), see Figure 7.1 for an illustration.

Lemma 7.4. *Up to relabeling of G_i, we have the following:*

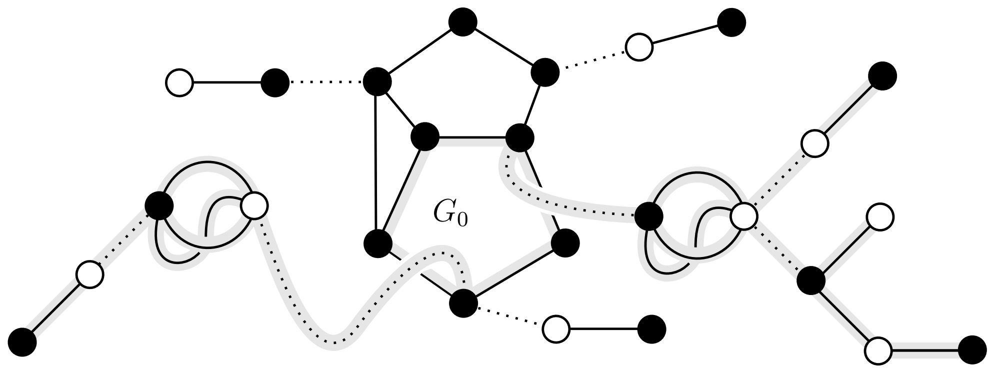


Figure 7.1: A degeneration of a ribbon graph G in $\mathcal{RG}^{\underline{k},*}_{g,n}$ into 9 components. The zero-weight static bridges are dotted. The unique non-bipartite component G_0 has 4 faces of degrees $7, 5, 5, 3$. Two "branches" are glued to the corners of one of the faces of degree 5. Their boundaries (in grey) form together a face of the initial graph G.

- *G_0 is non-bipartite and $G_1, \ldots, G_m$ are bipartite;*
- *graphs $G_1, \ldots, G_m$ each have one face;*
- *there is a partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_m$ of $\{1, \ldots, r\}$ such that:*
  - *the labels of vertices in the bipartite graphs G_i, $i = 1, \ldots, m$ are $\bigcup_{p \in A_i} I_p$ for one part (which we color in black) and $\bigcup_{p \in A_i} J_p$ for another part (which we color in white);*
  - *the labels of vertices in G_0 are $I_0 \cup \bigcup_{p \in A_0} I_p \cup \bigcup_{p \in A_0} J_p$;*
- *each G_i is positive at $(L_i)_{i \in V(G_i)}$.*

Since for each $i = 1, \ldots, m$ the graph G_i has one face, each "branch" of the tree-like structure emanating from G_0 is also a graph with one face. Take now a face of G_0 and all the branches of the tree-structure that are glued to some corner of this face via a zero-length static edge. Together all these faces form one face of the initial graph G (see Figure 7.1). For each $i = 1, \ldots, m$, let the unique face of G_i be part of the the face of G of degree $k_{a(i)}$. We call $a : \{1, \ldots, m\} \to \{1, \ldots, s\}$ the *attachment map*. Then for each $i = 1, \ldots, s$, the graph G_0 has a face of (odd) degree

$$k_i^0 = k_i - \sum_{j:\ a(j)=i} (2|E(G_j)| + 2). \tag{7.4}$$

Remark 7.5. *Note that $|E(G_i)| \geq 1$. Hence, in order for k_i^0 to be positive, we need $k_i \geq 5$. In particular, if $k_i \leq 3$ for all i, no degenerations of ribbon graphs in $\mathcal{RG}_{g,n}^{\underline{k},*}$ occur on the walls. This explains why the expression for the top-degree term of the counting functions for trivalent ribbon graphs as a Kontsevich polynomial (Theorem 1.8) is valid on the walls as well.*

Reverse construction

Now we would like to compute the sum of the top-degree terms on C of the contributions of all possible degenerations. For this, first choose:

- $g_0, \ldots, g_m \geq 0$ such that $g_0 + \ldots + g_m = g$,
- a partition $A_0 \sqcup A_1 \sqcup \ldots \sqcup A_m$ of $\{1, \ldots, r\}$,
- and an attachment function $a : \{1, \ldots, m\} \to \{1, \ldots, s\}$,

such that for all $i = 1, \ldots, s$, the value of k_i^0 given by (7.4) is positive.

Denote

$$n_0 = |I_0| + \sum_{p \in A_0} (|I_p| + |J_p|),\ n_i^+ = \sum_{p \in A_i} |I_p|,\ n_i^- = \sum_{p \in A_i} |J_p|.$$

Then choose the ribbon graphs $G_0, G_1, \ldots, G_m$ where

- for each $i = 1, \ldots, m$
  $$G_i \in \mathcal{E}^*_{g_i, n_i^+, n_i^-}$$
  is a vertex-bicolored ribbon graph with one face, with black vertex labels in $\bigcup_{p \in A_i} I_p$ and white vertex labels in $\bigcup_{p \in A_i} J_p$, and which is positive at $(L_i)_{i \in V(G_i)}$;
- $G_0 \in \mathcal{RG}_{g_0,n_0}^{\underline{k}^0,*}$, where $\underline{k}^0 = [k_1^0, \ldots, k_s^0]$, with vertex labels in $I_0 \cup \bigcup_{p \in A_0} I_p \cup \bigcup_{p \in A_0} J_p$, and which is positive at $(L_i)_{i \in V(G_0)}$.

The following Proposition is the key to the proof of Theorem 7.3. Its proof is given in section 7.2.2.

Proposition 7.6. *Let C_0 be a highest-dimensional cell of $\mathcal{PS}_n$ adjacent to C which contains a path $L(\varepsilon)$, $\varepsilon \in (0, 1]$ such that $\lim_{\varepsilon \to 0} L(\varepsilon) \in C$ and*

$$\sum_{i \in I_j} L_i(\varepsilon) - \sum_{i \in J_j} L_i(\varepsilon) = 2^j \varepsilon,\ j = 1, \ldots, m.$$

Then the number of of ribbon graphs $G \in \mathcal{RG}_{g,n}^{\underline{k},*}$ which are positive on C_0 and which degenerate into $G_0 \sqcup G_1 \sqcup \ldots \sqcup G_m$ is equal to

$$\prod_{i=1}^{s} \left(k_i^0 \cdot \prod_{j \in a^{-1}(i)} |E(G_j)| \cdot \frac{(2\sigma_i + k_i^0 + 2(|a^{-1}(i)| - 1))!!}{(2\sigma_i + k_i^0)!!} \right), \tag{7.5}$$

where $\sigma_i = \sum_{j \in a^{-1}(i)} |E(G_j)|$.

The most important feature of the formula (7.5) is that it only depends on the combinatorial data $g_0, \ldots, g_m$, $A_0, \ldots, A_m$ and a. It does not depend on the particular choice of the graphs $G_0, \ldots, G_m$ or the particular choice of the cell C.

In particular, it allows us to compute the sum of the top-degree terms of the contributions of degenerations with combinatorial data $g_0, \ldots, g_m$, $A_0, \ldots, A_m$ and a as

$$top\left(\mathcal{N}_{g_0,n_0}^{\underline{k}^0}\right) \cdot \prod_{i=1}^{m} top\left(\mathcal{P}_{n_i^+,n_i^-}^{g_i}\right) \tag{7.6}$$

times the coefficient in (7.5). Each term should be evaluated at the corresponding vertex perimeters and the top-degree terms should be taken on the corresponding linear subspaces (we omit the techincal details).

Summing these products over all admissible combinatorial data $g_0, \ldots, g_m$, $A_0, \ldots, A_m$ and a, we obtain the difference between the top-degree term of $\mathcal{N}_{g,n}^{\underline{k}}$ on C_0 and C.

Notice that the top-degree terms of $\mathcal{P}_{n_i^+,n_i^-}^{g_i}$ were computed in Chapter 3 and are polynomial (Theorem 3.2). Moreover, both $\underline{k}^0$ and n_0 in (7.6) are strictly smaller then the initial parameters $\underline{k}$ and n, so the top-degree terms of $\mathcal{N}_{g_0,n_0}^{\underline{k}^0}$ can be assumed to be polynomial by induction hypothesis.

Since $top_{C_0}\left(\mathcal{N}_{g,n}^{\underline{k}}\right)$ is the generalized Kontsevich polynomial, this gives a polynomial expression for $top_C\left(\mathcal{N}_{g,n}^{\underline{k}}\right)$ which is independent of the choice of C. This concludes the proof of Theorem 7.3.

7.1.3 Application to odd strata of quadratic differentials

In this section we describe how Theorem 7.3 and the elements of its proof (section 7.1.2) are applied in my joint work with Duryev and Goujard about the computations of the volumes of odd strata of quadratic differentials.

Recall from section 1.3.4 the strategy of [DGZZ21] for expressing the volume of the principal stratum of meromorphic quadratic differentials of genus g with n simple poles $\mathcal{Q}(1^{4g-4+n}, -1^n)$ as a sum of contributions over stable graphs. The aim of the project is to apply the same strategy to an arbitrary stratum of meromorphic quadratic differentials with odd orders of zeroes and at most simple poles (we call such strata *odd*).

Fix $g \geq 0$ and an unordered partition $\underline{\kappa} = [\kappa_1, \ldots, \kappa_s]$ of $4g-4$, where all $\kappa_i \geq -1$ are odd. Consider the stratum $\mathcal{Q}(\underline{\kappa})$ of meromorphic quadratic differentials with zeroes (and poles) of orders κ_i. The zeroes and poles are unlabeled (but this point is of minor importance in our exposition here).

As before, any square-tiled surface S in $\mathcal{Q}(\underline{\kappa})$ admits a decomposition into maximal horizontal cylinders glued according to a certain stable graph Γ. Each edge e of Γ corresponds to a maximal horizontal cylinder in S with circumference $L_e \in \mathbb{Z}_{>0}$ and height $h_e \in \mathbb{Z}_{>0}$. Each vertex v of Γ corresponds to a metric ribbon graph in S composed of several conical singularities joined by horizontal saddle connections. Let g_v be its genus, n_v be the number of its boundary components, $\underline{k}_v$ be the partition representing the degrees of its vertices, and let L_v be the n_v-tuple of the perimeters of its boundary components (these are the circumferences of the cylinders glued to this ribbon graph).

Recall that a zero (or pole) of order κ_i of a quadratic differential corresponds to a conical singularity of angle $(\kappa_i + 2)\pi$ in the induced flat metric. Hence, each zero (or pole) of order κ_i gives rise to a vertex of degree $k_i = \kappa_i + 2$ in one of the metric ribbon graphs. Since $\kappa_i \geq -1$ is odd, $k_i \geq 1$ is also odd. To count square-tiled surfaces in $\mathcal{Q}(\underline{\kappa})$ whose cylinder decomposition corresponds to a stable graph Γ, we are thus naturally led to the counting functions for metric ribbon graphs with odd degrees of vertices $\mathcal{N}^{\underline{k}_v}_{g_v,n_v},\ v \in V(\Gamma)$.

We can now write down the number of square-tiled surfaces in $\mathcal{Q}(\underline{\kappa})$ with at most N squares and corresponding to a stable graph Γ as

$$c_\Gamma \cdot \sum_{\sum_{e \in E(\Gamma)} L_e \cdot h_e \leq N} \prod_{e \in E(\Gamma)} L_e \cdot \prod_{v \in V(\Gamma)} \mathcal{N}^{\underline{k}_v}_{g_v,n_v}(L_v), \tag{7.7}$$

where c_Γ is a certain normalizing constant that we do not make explicit here. Taking the leading term of the $N \to \infty$ asymptotics of this sum, we get the contribution of the stable graph Γ to the volume of $\mathcal{Q}(\underline{\kappa})$.

As before, in order to compute the asymptotics of (7.7), we are inclined to replace the counting functions $\mathcal{N}^{\underline{k}_v}_{g_v,n_v}$ by their top-degree terms, the generalized Kontsevich polynomials (Theorem 7.2). However, there is a subtlety. If the stable graph Γ contains at least one loop, based at some vertex v, then in the sum (7.7) the corresponding counting function $\mathcal{N}^{\underline{k}_v}_{g_v,n_v}$ is *always* evaluated on some subspace of the form $L_i = L_j$. But the expression for the top-degree term of the counting function as a generalized Kontsevich polynomial is in general not valid on such walls.

As explained in the previous section, on the walls the top-degree of the counting functions decreases by the sum of contributions of certain degenerated ribbon graphs. Thus, when replacing in (7.7) the counting functions by the generalized Kontsevich polynomials, we overcount by the sum of contributions of such degenerations.

One can show that, when taking the leading term of the $N \to \infty$ asymptotics, this corresponds to adding to $\operatorname{Vol} \mathcal{Q}(\underline{\kappa})$ some products of volumes of certain smaller-dimensional odd strata of quadratic differentials and volumes of minimal strata of Abelian differentials, with certain weights. We call this expression the "completed volume" of $\mathcal{Q}(\underline{\kappa})$.

In the notations of the previous section, the smaller-dimensional odd strata of quadratic differentials correspond to the non-bipartite ribbon graphs G_0, while the minimal strata correspond to the bipartite ribbon graphs with one face G_i, $i \geq 1$. The weight of each product of volumes can be computed by using the degeneration coefficients of Proposition 7.6.

7.1.4 Generalized Kontsevich polynomials

We give here the definition of the generalized Kontsevich polynomials.

Recall the construction of the combinatorial moduli space $\mathcal{M}_{g,n}^{comb}$ from section 1.2.5. It is a cellular complex whose points correspond to metric ribbon graphs of genus g with n labeled faces and all vertices of degree at least 3. The highest dimensional cells of this complex correspond to trivalent metric ribbon graphs. Passing to a codimension d cell amounts to a contraction of d edges in some trivalent graph, and thus to graphs with vertices of degree larger than 3.

Fix now $g \geq 0$, $n \geq 1$ and an odd partition $\underline{k} = [1^{m_0}, 3^{m_1}, \ldots]$ satisfying (7.1). Denote $m_* = (m_0, m_1, m_2, \ldots)$ the infinite sequence of non-negative integers m_i, almost all zero. Let $\mathcal{M}_{m_*,n}^{comb}$ be the combinatorial moduli (orbi)space of metric ribbon graphs of genus g, with n labeled boundary components and with m_i vertices of degree $2i+1$ (the construction is analogous to the construction of $\mathcal{M}_{g,n}^{comb}$). If $m_0 = 0$, $\mathcal{M}_{m_*,n}^{comb}$ is naturally a subspace of $\mathcal{M}_{g,n}^{comb}$ and is a union of cells of real codimension $2M$, where

$$M = \sum_{i=1}^{\infty} m_i(i-1). \tag{7.8}$$

If $m_0 \neq 0$, the Strebel construction still gives a map $\mathcal{M}_{m_*,n}^{comb} \to \mathcal{M}_{g,n} \times \mathbb{R}_{>0}^n$ and we still have $\dim \mathcal{M}_{m_*,n}^{comb} = \dim \mathcal{M}_{g,n} - 2M$.

In any case the classes ψ_i can be pulled back to $\mathcal{M}_{m_*,n}^{comb}$ and we define

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{m_*} = \int_{\mathcal{M}^{comb}_{m_*,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \times [\mathbb{R}^n_{>0}],$$

where $[\mathbb{R}^n_{>0}]$ stands for the fundamental class with compact support of $\mathbb{R}^n_{>0}$.

There is a natural orientation on each component of $\mathcal{M}^{comb}_{m_*,n}$ [Kon92]. If $m_0 = 0$, it can be seen that, with this orientation, $\mathcal{M}^{comb}_{m_*,n}$ is a cycle with non-compact support in a certain compactification $\overline{\mathcal{M}}^{comb}_{g,n}$ and defines a homology class $W_{m_*,n} \in H_{6g-6+2n-2M}(\overline{\mathcal{M}}'_{g,n}; \mathbb{Q})$, where $\overline{\mathcal{M}}'_{g,n}$ is a certain compactification of $\mathcal{M}_{g,n}$, see [AC96] for details. In this case one can also write

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{m_*} = \int_{W_{m_*,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

Finally, for $\underline{k} = [1^{m_0}, 3^{m_1}, \ldots]$, the *generalized Kontsevich polynomial* $N^{\underline{k}}_{g,n}$ is defined as

$$N^{\underline{k}}_{g,n}(L) = \frac{1}{2^{5g-6+2n-2M}} \sum_{\sum d_i = 3g-3+n-M} \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{m_*}}{d_1! \cdots d_n!} L_1^{2d_1} \cdots L_n^{2d_n}.$$

The intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{m_*}$ where conjectured by Kontsevich to be polynomials in the usual intersections of ψ classes, which was proved in [DFIZ93]. It was later conjectured [AC96] and proved [Mon04] that the identity also holds on the level of cohomology classes: combinatorial classes $W_{m_*,n}$ are expressible in terms of the algebro-geometric ones.

7.2 Proofs

7.2.1 Proof of Lemma 7.4

Proof of Lemma 7.4. Recall that for any static bridge e one component of $G - e$ is bipartite and the other is not. It means that among the G_i there is exactly one non-bipartite ribbon graph (if there were two of them, any static bridge separating them would violate the above property). We assume that G_0 is non-bipartite and $G_1, \ldots, G_m$ are bipartite.

We now prove that each bipartite graph G_i has one face. Consider a graph G_i which is at a "leaf" of the tree-like structure. The unique zero-length static edge incident to G_i is incident to a corner of some face of G_i. If G_i had another face, it would also be a face of G, thus of odd degree, which is impossible since G_i is bipartite. Hence G_i has one face and we can remove it from G with the adjacent zero-length static edge. This decreases the degree of some face of G by an even number and so this face will still be of odd degree. We can continue removing the bipartite graphs G_i in the same manner.

To prove the third property, recall the formula for the weight of a static bridge e (Lemma 2.10). On C this weight is zero, which implies a linear relation on the L_i of the form $\sum_{i \in I} L_i = \sum_{j \in J} L_j$, where I, J are the labels of vertices in the two parts of the bipartite component of $G - e$. But the only possible relations of this form are (7.3) and the sums thereof. This implies that for each bipartite G_i, the labels of vertices in the two parts must be of the form $I_{i_1} \cup \ldots \cup I_{i_p}$ and $J_{i_1} \cup \ldots \cup J_{i_p}$, which implies the third property.

Finally, the fourth property is necessary for the contribution of the corresponding graph to be non-zero on C (see section 2.2.5). □

7.2.2 Proof of Proposition 7.6

Fix the combinatorial data $g_0, \ldots, g_m$, $A_0, \ldots, A_m$, a and the ribbon graphs $G_0, \ldots, G_m$ as in section 7.1.2. We have to count the number of ways to join together into a tree-like structure the graphs G_i with m bridges, in such a way as to produce a ribbon graph positive on C_0. We call such joining *admissible*.

Note that the graphs $G_1, \ldots, G_m$ are vertex-bicolored, but the joining bridges might join vertices of the same color.

Lemma 7.7. *A joining is admissible if and only if the weights of all the joining bridges are positive at $L(\varepsilon)$ for $\varepsilon > 0$.*

Proof. This is clearly a necessary condition for G to be positive at $L(\varepsilon)$ for $\varepsilon > 0$. It is also sufficient, because, denoting $L|_i(0) := (L_j(0))_{j \in V(G_i)}$, we have

$$\mathrm{Vol}\, M_G(L(0)) = \prod_{i=1}^{m} \mathrm{Vol}\, M_{G_i}(L|_i(0)) > 0,$$

and so $\mathrm{Vol}\, M_G(L(\varepsilon)) > 0$ for small values of ε by continuity. □

Note that since the attachment map a is fixed, we know which of the graphs G_i are in the branches of the tree-like structure which are joined to the face of G_0 of degree k_i^0. It means that we can count separately for each $i = 1, \ldots, s$, the number of admissible joinings of the face of G_0 of degree k_i^0 with the corresponding graphs, and than take the product over i. This explains the product structure of the coefficient (7.5).

Hence, it is sufficient to consider the case $i = 1$. Without loss of generality, assume that $a^{-1}(1) = \{1, 2, \ldots, m'\}$, for some $m' \geq 1$. Denote $e_i = |E(G_i)|$. We will prove that the number of admissible joinings of $G_0, G_1, \ldots, G_{m'}$, where the joining bridges adjacent to G_0 are only adjacent to its face of degree k_1^0, is equal to

$$k_1^0 \cdot \prod_{i=1}^{m'} e_i \cdot \frac{(2\sigma_1 + k_1^0 + 2(m'-1))!!}{(2\sigma_1 + k_1^0)!!},$$

where $\sigma_1 = \sum_{i=1}^{m'} e_i$.

For $i = 1, \dots, m'$, denote by $D_i(L)$ the linear function of L which is the difference between the sums of black and white vertex perimeters of the graph G_i. Then

$$D_i(L(\varepsilon)) = \sum_{p \in A_i} \left(\sum_{j \in I_p} L_j(\varepsilon) - \sum_{j \in J_p} L_j(\varepsilon) \right) = \sum_{p \in A_i} 2^p \varepsilon.$$

Up to relabeling we can assume $\max A_1 < \max A_2 < \ldots < \max A_{m'}$.

Lemma 7.8. *For each $i = 2, \dots, m'$ we have for $\varepsilon > 0$*

$$D_i(L(\varepsilon)) > D_{i-1}(L(\varepsilon)) + \ldots + D_1(L(\varepsilon)).$$

Proof. On the one hand, $D_i(L(\varepsilon)) \geq 2^{\max A_i}\varepsilon$. On the other hand,

$$\begin{aligned} D_{i-1}(L(\varepsilon)) + \ldots + D_1(L(\varepsilon)) &= \sum_{p \in \bigcup_{j=1}^{i-1} A_j} 2^p \varepsilon \\ &\leq (1 + 2^1 + 2^2 + \ldots + 2^{\max A_{i-1}})\varepsilon < 2^{\max A_{i-1}+1}\varepsilon \leq 2^{\max A_i}\varepsilon. \end{aligned}$$

□

To every possible joining of the G_i we associate a tree T on $m+1$ vertices labeled from 0 to m (the vertices correspond to the connected components G_i and the edges correspond to the joining bridges).

Root the tree T at the vertex 0. Then every vertex $i \geq 1$ has a well-defined parent vertex and descendant vertices (those for which the unique path to the root passes through this vertex). We will also say that G_i is a parent/descendant of G_j, if i is a parent/descendant of j in T.

Lemma 7.9. *A joining is admissible if and only if for every $i = 1, \dots, m$ the following holds:*

- *if all of the descendants of G_i have labels smaller then i, then the bridge joining G_i to its parent is in the black corner of G_i;*

- *otherwise, the bridge joining G_i to its parent and the bridge joining G_i to the subtree containing the descendant of G_i of maximal label are in the corners of G_i of different colors.*

Proof. In the first case, let s_p be 1 if the bridge joining G_i to its parent is in the black corner of G_i, and -1 otherwise. Then, by Lemma 2.10, the weight of this bridge at $L(\varepsilon)$ is equal to

$$s_p D_i(L(\varepsilon)) + \sum_j \pm D_j(L(\varepsilon)),$$

where the sum is over the labels j of all descendants of G_i.

Note that $j < i$ for all descendants G_j of G_i. Hence, if $s_p = 1$, this is at least $D_i(L(\varepsilon)) - \sum_j D_j(L(\varepsilon)) > 0$ when $\varepsilon > 0$, by Lemma 7.8. On the contrary, if $s_p = -1$, this is at most $-D_i(L(\varepsilon)) + \sum_j +D_j(L(\varepsilon)) < 0$ when $\varepsilon > 0$, again by Lemma 7.8. Hence the weight of this bridge is positive for $\varepsilon > 0$ if and only if $s_p = 1$, as desired.

In the second case, let i' be the descendant of i of maximal label and let $i' = i_0, i_1, \ldots, i_r = i$ be the vertices on the unique path from i' to i. Let $s_p(i_j)$ be 1 if the bridge joining G_{i_j} to $G_{i_{j+1}}$ is in the black corner of G_{i_j}, and -1 otherwise. Similarly, let $s_d(i_j)$ be 1 if the bridge joining G_{i_j} to $G_{i_{j-1}}$ is in the black corner of G_{i_j}, and -1 otherwise.

Note that the label $i' = i_0$ is bigger than the labels of all of its descendants, so by the first case $s_p(i_0) = 1$. Then for each $1 \le j \le r$ the length at $L(\varepsilon)$ of the bridge joining G_{i_j} and $G_{i_{j+1}}$ is equal to

$$\left(\prod_{k=1}^{j} (-1) \cdot s_p(i_k) \cdot s_d(i_k)\right) \cdot D_{i'}(L(\varepsilon)) + \sum_k \pm D_k(L(\varepsilon)), \tag{7.9}$$

where the sum is over the labels k equal to i_j and to all of descendants of i_j except $i_0 = i'$.

Again, $k < i'$ for all k, so by the same reasoning as above, the weight of this bridge is positive for $\varepsilon > 0$ if and only if the product in (7.9) is equal to 1. We conclude that $\prod_{k=1}^{j} (-1) \cdot s_p(i_k) \cdot s_d(i_k) = 1$ for all $1 \le j \le r$, and so $s_p(i_j) s_d(i_j) = -1$ for all $1 \le j \le r$. In particular, $s_p(i_r) s_d(i_r) = s_p(i) s_d(i) = -1$, as desired.

□

We now establish a bijection between admissible joinings and certain sequences of markers in the corners the ribbon graphs G_i. Each marker will correspond to a place were one of the joining bridges is glued. Thus each joining bridge produces two markers. If there are several markers in the same corner of G_i, their relative order around the vertex is part of the data.

For a marker a, denote by $l(a)$ the label of the component in which a is contained. If $l(a) > 0$, let $s(a)$ be 1 if a is contained in a black corner, and -1 otherwise.

Lemma 7.10. *There is a bijection between admissible joinings and sequences $(a_1, b_1, \dots, a_{m'}, b_{m'})$ of $2m'$ markers in the corners of G_i satisfying the following conditions:*

- $l(a_1) = 0$*;*
- *for all* $1 \le i \le m'$*, either*

$$\begin{cases} l(a_i) \in \{0, l(a_1), l(b_1), \dots, l(a_{i-1}), l(b_{i-1})\}, \\ l(b_i) = \max\{0, l(a_1), l(b_1), \dots, l(a_{i-1}), l(b_{i-1})\}^c, \\ s(b_i) = 1, \end{cases} \tag{7.10}$$

or

$$\begin{cases} l(a_i) = l(b_i) \notin \{0, l(a_1), l(b_1), \dots, l(a_{i-1}), l(b_{i-1})\}, \\ s(a_i) \neq s(b_i), \end{cases} \tag{7.11}$$

where the superscript c stands for the complement in $\{0, 1, \dots, m'\}$.

Proof. The sequence of markers corresponding to an admissible joining consists of the $2m'$ places where the joining bridges are glued to the G_i, written down in a particular order, which we describe in Algorithm 1. In this algorithm S is interpreted as the set of *non-visited* components. Note that $0 \notin S$ from the start, so component 0 is considered to be visited from the start. When the algorithm terminates, res contains the corresponding sequence of markers. See Figure 7.2 for an example computation.

The algorithm terminates since at each step of the while-loop we remove from S at least one element $(\max S)$.

In words, the Algorithm 1 traverses the tree T corresponding to the joining by starting with the 0-component, then at each iteration of the while-loop it appends to the already traversed subtree the path joining it to the not yet visited component of maximal label. It first writes in res the marker at a place where this path is glued (with a joining bridge) to the already traversed subtree. Then, for each edge along this path, starting from the not yet visited component of maximal label, it appends to res the markers of the joining bridge corresponding to this edge. In particular, it is clear that all of the

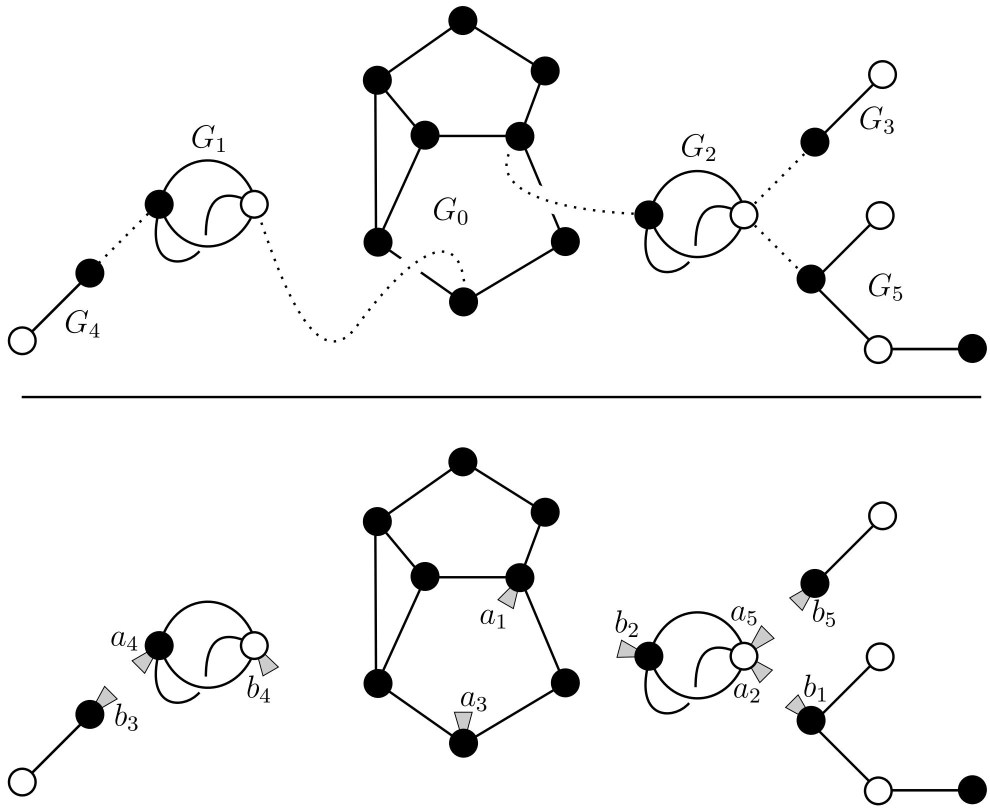


Figure 7.2: An admissible joining (top) and the corresponding sequence of markers (bottom) produced by Algorithm 1.

Algorithm 1 From an admissible joining to a sequence of markers

1: $S := \{1, \dots, m'\}$
2: $res := ()$
3: **while** $S \neq \varnothing$ **do**
4: $\quad \gamma = (i_0 = \max S, i_1, \dots, i_r = 0) :=$ path from $\max S$ to 0 in T
5: $\quad i_{r'} :=$ first component in γ with label not in S (i.e. already visited)
6: $\quad$ append to res a marker at a place where the bridge joining $G_{i_{r'-1}}$ and $G_{i_{r'}}$ is glued to $G_{i_{r'}}$
7: $\quad$ **for** $j = 0, \dots, r' - 1$ **do**
8: $\qquad$ append to res a marker at a place where the bridge joining G_{i_j} and $G_{i_{j+1}}$ is glued to G_{i_j}
9: $\qquad$ append to res a marker at a place where the bridge joining G_{i_j} and $G_{i_{j+1}}$ is glued to $G_{i_{j+1}}$
10: $\quad$ **end for**
11: $\quad$ append to res a marker at a place where the bridge joining $G_{i_{r'-1}}$ and $G_{i_{r'}}$ is glued to $G_{i_{r'-1}}$
12: $\quad$ remove from S the components $i_0, \dots, i_{r'-1}$
13: **end while**

edges of T will be traversed, and so the resulting sequence res will contain all of the $2m'$ markers corresponding to the $2m'$ joining bridges.

We now show that the sequence of markers thus constructed satisfies the conditions of the Lemma.

Clearly $l(a_1) = 0$. Let $a_i, b_i, \dots, a_{i+r}, b_{i+r}$ be a sequence of markers added to res during one iteration of the while-loop. a_i is the loop added in step 6 of the Algorithm. In particular, it is in one of the already visited components, so the first condition of (7.10) is satisfied for a_i. b_i is the first marker added in the for-loop 7. By design, it is in the not yet visited component of maximal label, so the second condition of (7.10) is satisfied for b_i. The label of the component of b_i is in particular bigger than the labels of all of its descendants, so by the first point of Lemma 7.9, $s(b_i) = 1$, which is the third point of (7.10).

$a_{i+1}, b_{i+1}, \dots, a_{i+r}, b_{i+r}$ are the rest of the markers added in steps 7 to 11 of the Algorithm. For each $j = 1, \dots, r$, a_{i+j} and b_{i+j} are markers in the corners of G_{i_j}, and so $l(a_{i+j}) = l(b_{i+j})$. Moreover, all of the components G_{i_j} are not yet visited, so the first condition in (7.11) is satisfied for all a_{i+j}, b_{i+j}. The second condition of (7.11) follows from the second point of Lemma 7.9, since for all $j = 1, \dots, r$, the component $\max S$ is the descendant of maximal label of the component with label $l(a_{i+j}) = l(b_{i+j})$ (because all of its descendants are in S at this stage of the Algorithm).

Finally, we have to show that Algorithm 1 establishes a bijection between admissible joinings and sequences of markers satisfying conditions of the Lemma. We do this by describing the inverse algorithm.

To reconstruct the joining from the sequence of markers $(a_1, b_1, \dots, a_{m'}, b_{m'})$ it is enough to find which pairs of markers should be joined by a joining bridge. Subdivide the sequence of markers into intervals of the form $(a_i, b_i, \dots, a_{i+r}, b_{i+r})$ with the pair (a_i, b_i) satisfying condition (7.10) and the pairs $(a_{i+1}, b_{i+1}), \dots, (a_{i+r}, b_{i+r})$ satisfying condition (7.11) (such subdivision is clearly unique). Then treat the intervals from left to right by gluing (for each interval) b_i with a_{i+1}, b_{i+1} with a_{i+2}, ..., b_{i+r-1} with a_{i+r}, and finally b_{i+r} with a_i (which joins this branch with what has been constructed from the previous intervals). It is straightforward to check that the two algorithms are inverses of each other. □

Before we conclude the proof of Proposition 7.6, we need the following elementary counting lemma.

Lemma 7.11. *Let $n \geq 1$ and $d_1, \dots, d_n \geq 1$. Suppose that in a rooted tree T any path from the root to a leaf passes successively through n vertices having $d_{\pi(1)}, \dots, d_{\pi(n)}$ children respectively, for some permutation π of the set $\{1, \dots, n\}$ depending on the leaf. Then T has $d_1 d_2 \cdots d_n$ leaves.*

Proof. The proof is by induction on n. For $n = 1$ the statement is trivial. For $n \geq 2$, suppose the root of T has d_k children for some $k \in \{1, \dots, n\}$. Each of the root subtrees satisfies the conditions of the lemma with parameters $\{d_i, i \neq k\}$ and so, by induction hypothesis, has $\prod_{1 \leq i \leq n, i \neq k} d_i$ leaves. Since there are d_k such subtrees, T has $d_1 d_2 \cdots d_n$ leaves in total, as desired. □

Conclusion of the proof of Proposition 7.6. It remains to count the sequences of loops satisfying the conditions of Lemma 7.10. We do this by successively choosing the markers a_i, b_i and keeping track of how many choices we are left with after each step. However, the number of choices we have at each step *depends on the choices we made before*, so we cannot directly apply the combinatorial product rule. Instead, we will represent all possible choices as a (rooted) decision tree (with leaves corresponding to the final sequences of markers) and then apply Lemma 7.11 to this tree.

Clearly, there are k_1^0 choices for the location of a_1 (since our branches are only glued to the face of G_0 of degree k_1^0). By (7.10) we necessarily have $l(b_1) = m'$ and $s(b_1) = 1$, for which there are $e_{m'}$ choices. At each next step $i = 2, \dots, m'$, we can freely choose the location of the marker a_i among all of the $k_1^0 + 2e_1 + \ldots + 2e_{m'} + 2(i-1)$ places available at this stage (before stage i we have already chosen the places for $a_1, b_1, \dots, a_{i-1}, b_{i-1}$, which creates $2(i-1)$ additional places).

Depending on the choice of a_i, there are two possibilities:

- if $l(a_i) \in \{0, l(a_1), l(b_1), \dots, l(a_{i-1}), l(b_{i-1})\}$, the $l(b_i)$ and $s(b_i)$ are now fixed by (7.10), and there are $e_{l(b_i)}$ choices for the place of b_i, since at this stage none of the markers has been placed in the component with label $l(b_i)$;
- if $l(a_i) \notin \{0, l(a_1), l(b_1), \dots, l(a_{i-1}), l(b_{i-1})\}$, the $l(b_i)$ and $s(b_i)$ are now fixed by (7.11), and there are $e_{l(b_i)}$ choices for the place of b_i, since at this stage only one marker (namely, a_i) has been placed in the component with label $l(b_i)$, but it was placed in a corner of color different to that of b_i, so this does not affect the number of choices.

When we have finally chosen all of the loops $a_1, b_1, \dots, a_m, b_m$, the numbers of choices we had along the way were equal to

$$k_1^0, e_{l(b_1)}, k_1^0 + 2\sigma_1 + 2, e_{l(b_2)}, k_1^0 + 2\sigma_1 + 4, e_{l(b_3)}, \dots, k_1^0 + 2\sigma_1 + 2(m' - 1), e_{l(b_{m'})}. \tag{7.12}$$

Note that $l(b_1), \dots, l(b_{m'})$ is a permutation of the set $\{1, \dots, m'\}$ (this follows from (7.10) and (7.11); $l(b_i)$ is always among the labels we have not used before). It means that our decision tree satisfies the conditions of Lemma 7.11 with parameters (7.12), and so it has

$$k_1^0 e_1 \cdots e_{m'} (k_1^0 + 2\sigma_1 + 2)(k_1^0 + 2\sigma_1 + 4) \dots (k_1^0 + 2\sigma_1 + 2(m' - 1))$$

leaves, which completes the proof. □

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