Polynomial counting functions for metric ribbon graphs

Ivan Yakovlev

Max Planck Institute for Mathematics, Bonn

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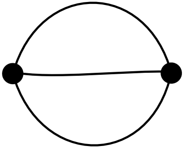
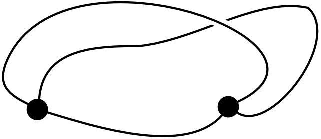
iyakovlev23.github.io

Ribbon graphs (= combinatorial maps)

A ribbon graph is a graph (loops and multiple edges allowed) with a circular ordering of (half-)edges incident to every vertex.





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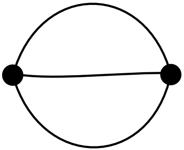

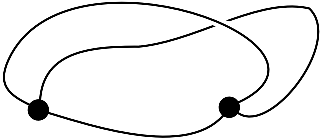

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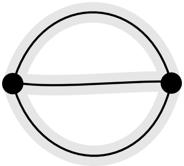

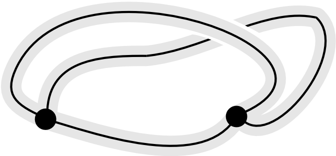

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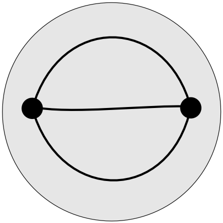

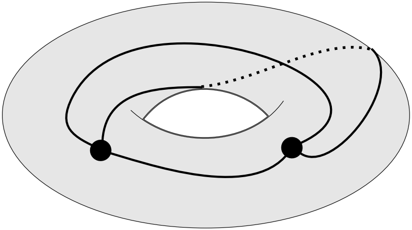

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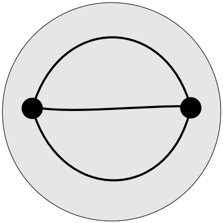

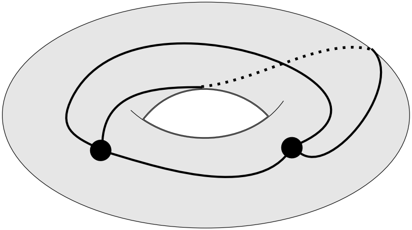

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Euler's formula: $|V(G)| - |E(G)| + |F(G)| = 2 - 2g$.

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$$\mathcal{N}_G(L_1, \ldots, L_n) = \# \left\{ \begin{array}{c} \textit{integer} \text{ metrics on } G \text{ with perimeter of} \\ \text{the } i\text{-th face equal to } L_i \end{array} \right\}$$

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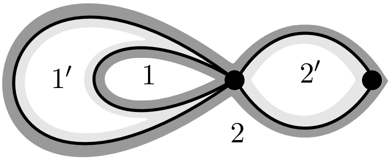
For a *face-bicolored* ribbon graph G with k black and l white faces, and $L_1, \ldots, L_k, L'_1, \ldots, L'_l \in \mathbb{Z}$, define analogously

$$\mathcal{N}_G(L_1, \ldots, L_k; L'_1, \ldots, L'_l).$$

Computing a counting function



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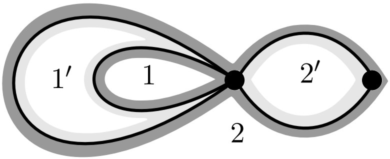

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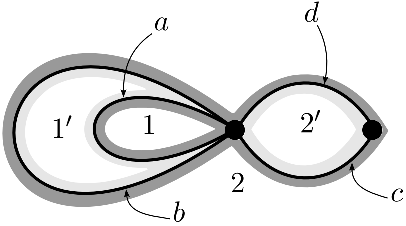

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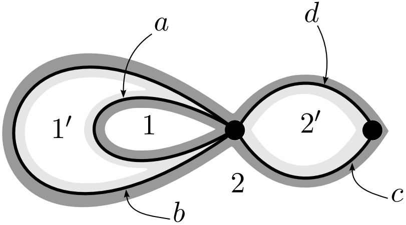

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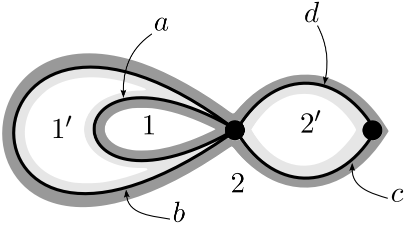

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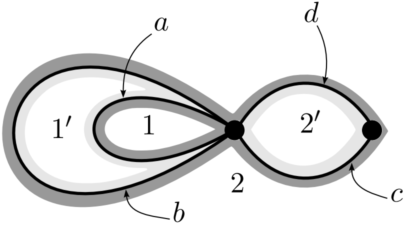

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$$\begin{cases} a,b,c,d > 0 \\ a = L_1 \\ b+c+d = L_2 \\ a+b = L'_1 \\ c+d = L'_2 \end{cases} \iff \begin{cases} a,b,c,d > 0 \\ a = L_1 \\ b = L'_1 - L_1 \\ c+d = L'_2 \end{cases} \Rightarrow \mathcal{N}_G(L;L') = \mathbf{1}_{L'_1 > L_1} \cdot (L'_2 - 1).$$

Counting functions

Proposition

For any G the function $\mathcal{N}_G$ is piecewise (quasi-)polynomial. The regions of polynomiality are cut out by a certain hyperplane arrangement $\mathcal{H}_n$ ($\mathcal{H}_{k,l}$).

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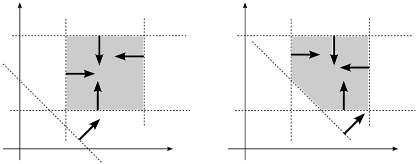

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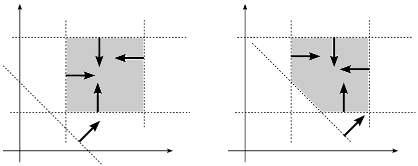

Note that the top-degree term of $\mathcal{N}_G$ gives the volume of this polytope!

Counting functions

Introduce the counting functions for families $\mathcal{RG}_{g,n}^{d}$ of ribbon graphs sharing the same genus g, number of faces n and vertex-degree profile d:

$$\mathcal{N}_{g,n}^{d}(L) = \sum_{G \in \mathcal{RG}_{g,n}^{d}} \frac{1}{|\operatorname{Aut}(G)|} \cdot \mathcal{N}_G(L).$$

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A priori piecewise (quasi-)polynomials...
But sometimes the top-degree term is polynomial!

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Consider the families of *trivalent* ribbon graphs: $d = [3, \ldots, 3] = [3^{4g+2n-4}]$.

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For any g, n, the top-degree term of $\mathcal{N}_{g,n}^{[3^{4g+2n-4}]}$ is a polynomial whose coefficients are the intersection numbers of ψ-classes on the moduli space of marked Riemann surfaces: $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$.

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Idea: polytopes corresponding to all graphs G can be glued together to form a space homeomorphic to $\overline{\mathcal{M}}_{g,n}$.

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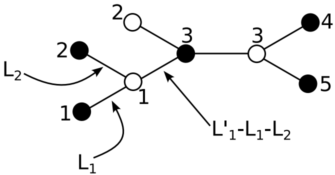
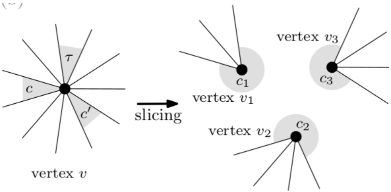
Idea: compute explicitly using a bijective approach!

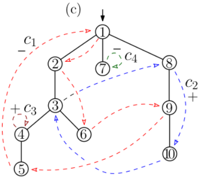
Example 2: one-vertex face-bicolored graphs

- $g = 0$: elementary count of metric plane trees;



- $g > 0$: reduce to case $g = 0$ using the bijection of Chapuy-Féray-Fusy '13 between 1-face maps and decorated plane trees, via vertex explosions.





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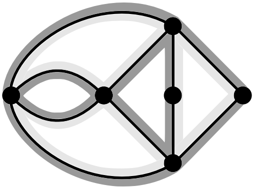

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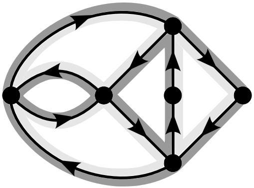

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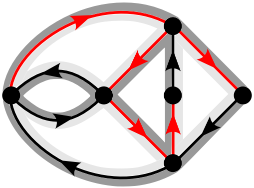

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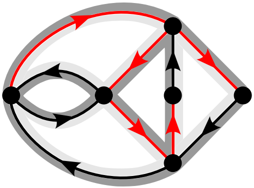

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For $g = 0$, the top-degree term of $\widetilde{\mathcal{N}}^{d}_{0,(k,l)}$ is a polynomial

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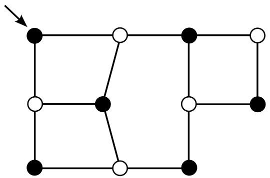

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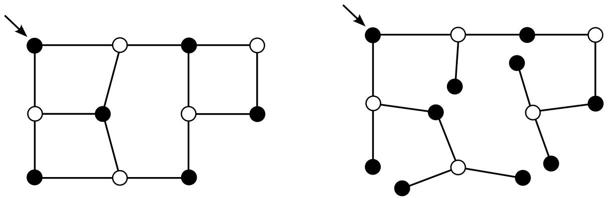

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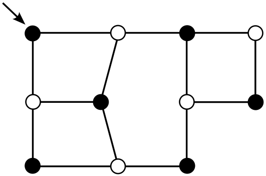

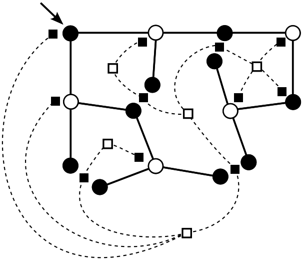

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- Formula $(k + l - 2)! \cdot (L_1 + \ldots + L_k)^{\ell(d)-1}$ suggests a double counting argument: each tree contributes $(L_1 + \ldots + L_k)^{\ell(d)-1}$.
- It would be interesting to have such an argument for trivalent graphs $\Rightarrow$ combinatorics of the numbers $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$.