# Metric ribbon graphs

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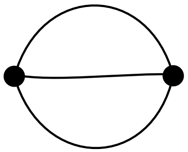

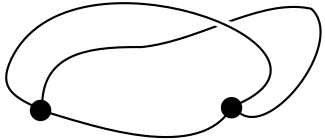

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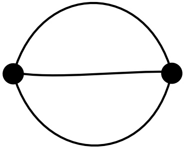

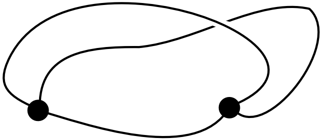

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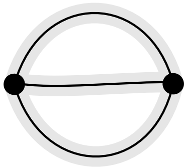

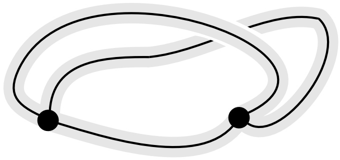

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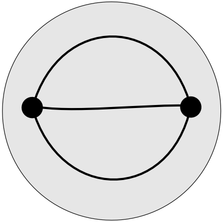

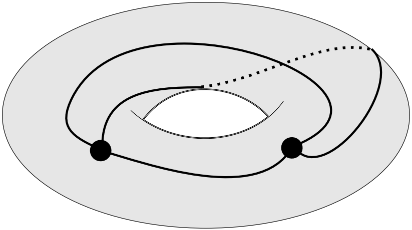

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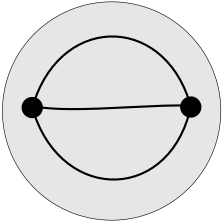

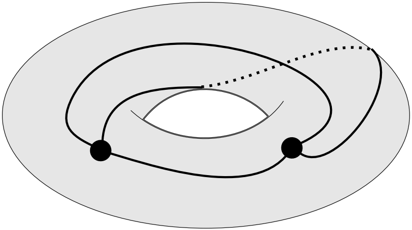

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Euler's formula: $|V(G)| - |E(G)| + |F(G)| = 2 - 2g$.

*Faces = boundary components.*

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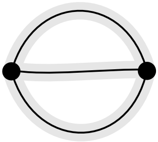

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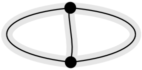

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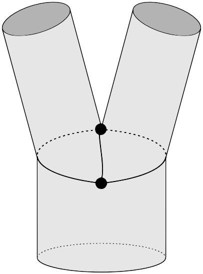

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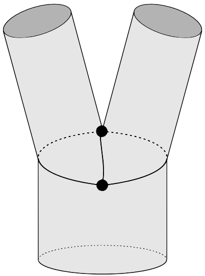

- zero/pole of order $k \geq -1 \leftrightarrow$ vertex of degree $k + 2$;
- poles of order 2 $\leftrightarrow$ infinite cylinders;
- residues at order 2 poles $\leftrightarrow$ perimeters of boundaries / cylinders.

# Families of ribbon graphs

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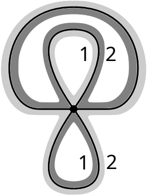

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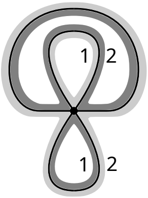


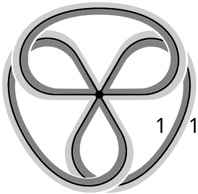

# Counting functions

For $L \in \mathbb{Z}^n$, let

$$\mathcal{P}_{g,n}(L) = \# \left\{ \begin{array}{c} \text{integer metric ribbon graphs of genus } g \\ \text{with } n \text{ boundaries of perimeters } L_1, \ldots, L_n \end{array} \right\}$$

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For $L \in \mathbb{Z}^k, L' \in \mathbb{Z}^l$, define analogously $\mathcal{P}_{g,(k,l)}(L, L')$.

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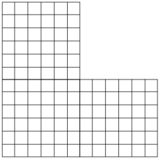

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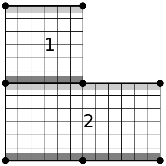

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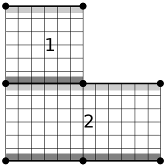


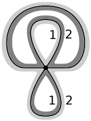

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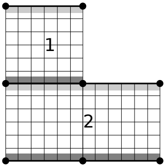


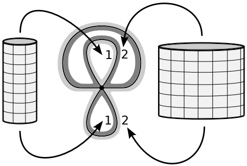

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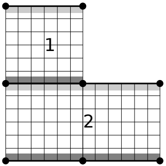


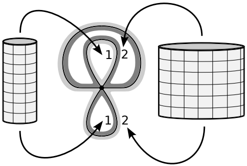


- connection to intersection theory on strata / moduli spaces.

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**Theorem (Kontsevich, '92)**

Up to lower order terms, $\mathcal{P}^3_{g,n}$ is a homogeneous polynomial

$$\frac{1}{2^{5g-6+2n}} \sum_{d_1+\ldots+d_n=3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} \frac{L_i^{2d_i}}{d_i!},$$

where $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$ are the intersection numbers of $\psi$-classes on the moduli space of marked Riemann surfaces.

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Witten's conjecture, $\mathcal{M}^{comb}_{g,n}$, Strebel differentials.

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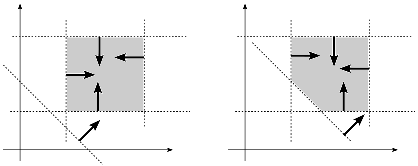

Counting integer points in a deforming polytope.

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## Theorem (Y. '23)

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- outside of the walls: **Okounkov, Pandharipande '06**; genus 0: **Gendron, Tahar '22**; **Chen, Prado '23** (intersection theory!).
- top-degree term on $\{L_i = L'_i,\ \forall i\} \Rightarrow$ cylinder contributions in $\mathcal{H}(2g-2)$.

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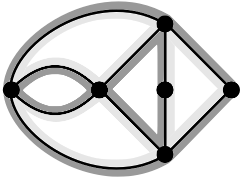

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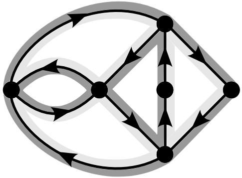

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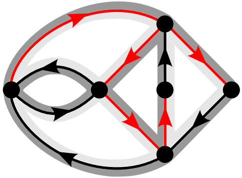

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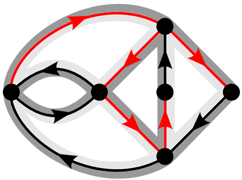

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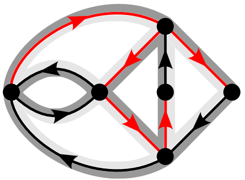

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Also true for $g > 0$...

# Ideas of proofs

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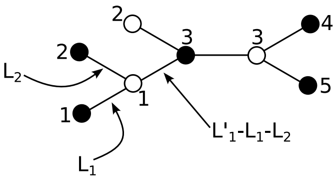

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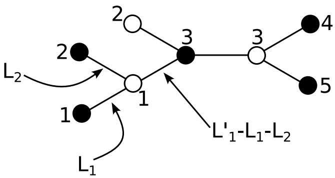
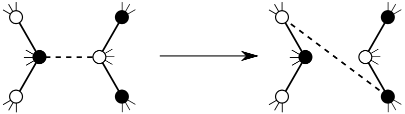
- when traversing a wall, we “loose” and “gain” an equal number of trees.



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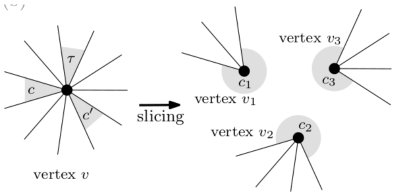


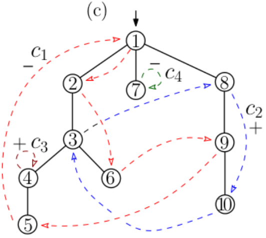

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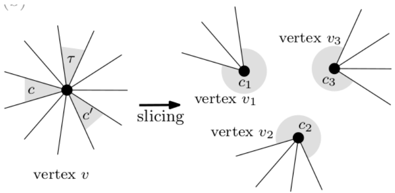
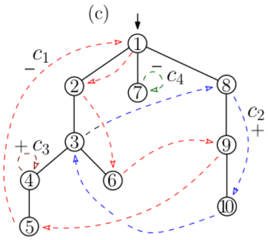
- allows to control the metric! □



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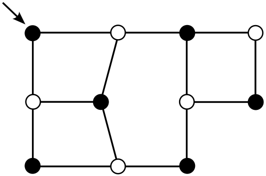

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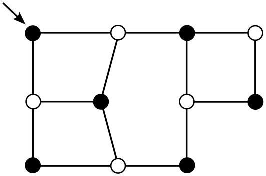

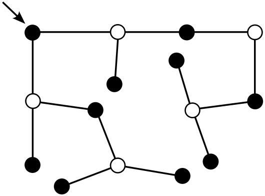

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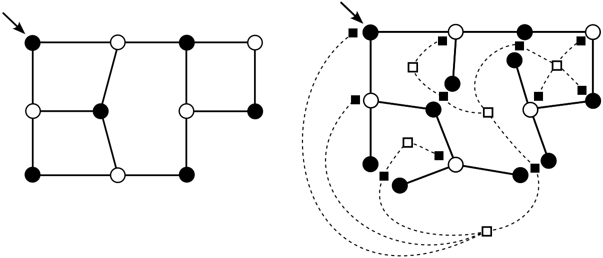

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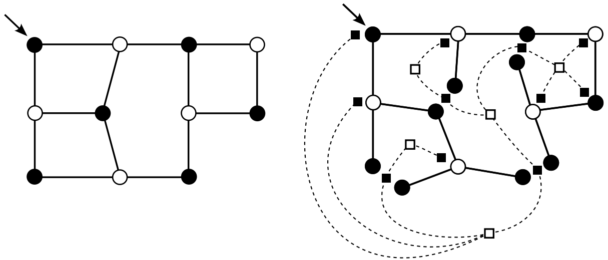
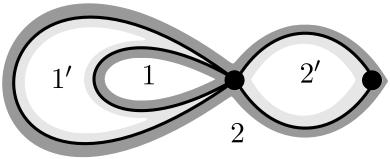


$\rightsquigarrow$ need to control both the metric and the combinatorics of a tree (positions of certain corners when going around the tree) $\rightsquigarrow$ *prefix-postfix sequences*. □

# Some open problems

- intersection theory interpretation of combinatorial results?
- understand the combinatorics of the Kontsevich's theorem;
- weighted counting functions for $g > 0$;
- applications to enumeration of square-tiled surfaces?

# Computing a counting function



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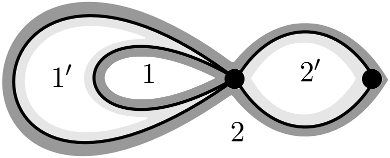


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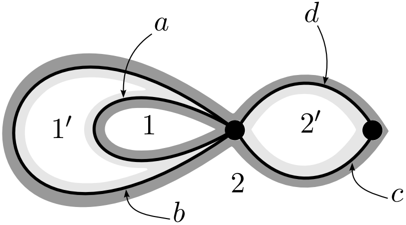


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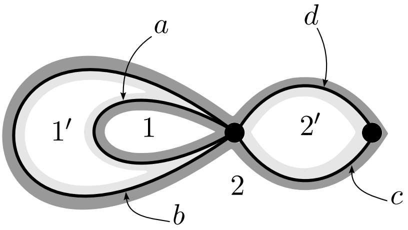


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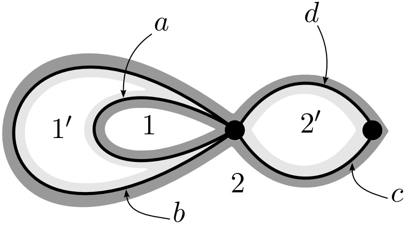


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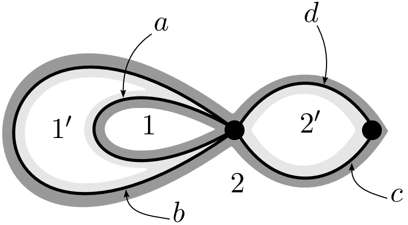


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# Piecewise polynomiality

Let $\mathcal{PS}_{k,l}$ be the polyhedral subdivision of $\mathbb{R}^k \times \mathbb{R}^l$ generated by the hyperplanes ("walls") of the form $\sum_{i\in I} L_i = \sum_{j\in J} L'_j,\ I \subset \{1,\dots,k\},\ J \subset \{1,\dots,l\}$.

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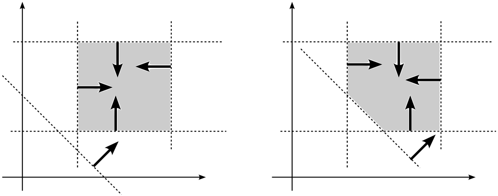
Idea of proof: counting integer points in a deforming polytope.



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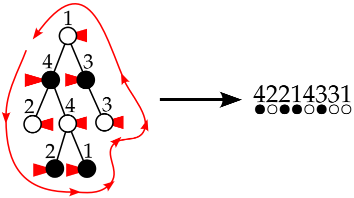
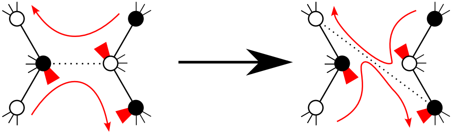
⇒ Prefix-postfix traversal!



# Flipping edges cleverly

For any edge, there is a unique way to flip it so that the prefix-postfix sequence does not change!



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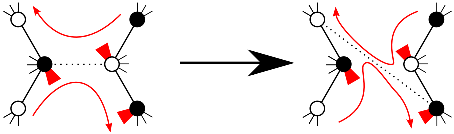

## Proposition (Y., '24)

*For any $k, l \geq 1$, any permutation of vertex labels $\pi$, and any point $(L; L')$ outside of the walls, there is exactly 1 tree with prefix-postfix sequence $\pi$ and contributing at $(L; L')$.*

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*Observation*: any tree contributes for values of $(L; L')$ in a *simplicial* cone

$$cone\left(e_i + e'_j : \bullet\!\overset{i}{-}\!\circ^j \text{ is an edge}\right) \subset (\mathbb{R}^k_+ \times \mathbb{R}^l_+) \cap \left\{\sum L_i = \sum L'_j\right\},$$

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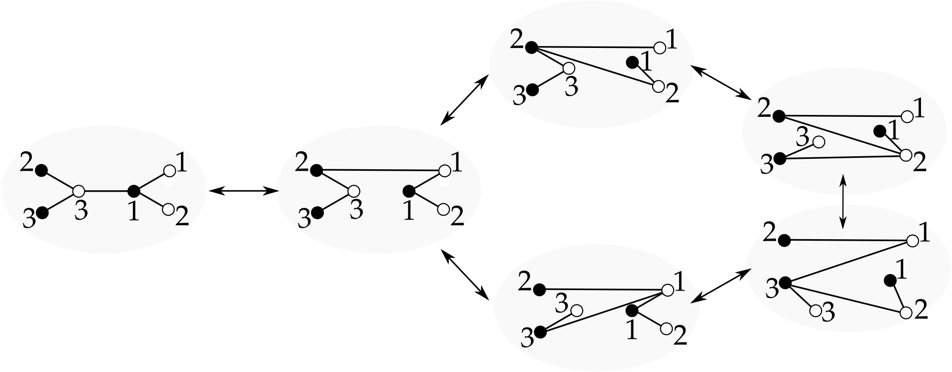
**Theorem (Y., '24)**

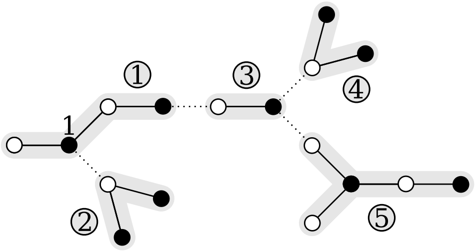
*For any $k, l \geq 1$ and any permutation $\pi$ of the vertex labels, the simplices corresponding to plane trees with prefix-postfix sequence $\pi$ form a triangulation of $\Delta_k \times \Delta_l$.*

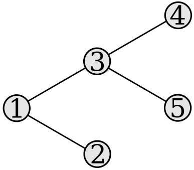
# Example of a triangulation

Example for $k = l = 3$:



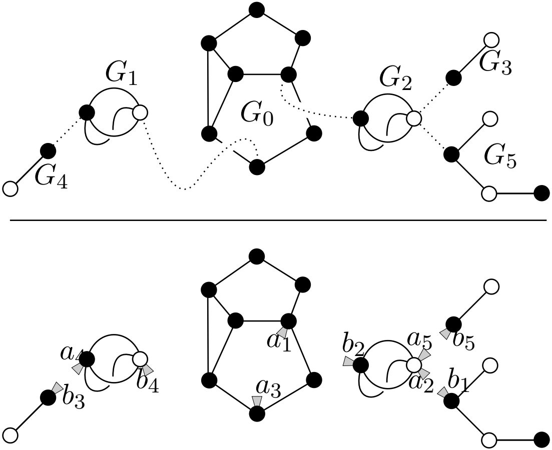
# Computing degeneration coefficients on the walls




$$(k+l-2)! = p^{b_1,\dots,b_n}_{w_1,\dots,w_n} + \sum_{t=2}^{n} \frac{(k+l-2)_{t-2}}{t!} \sum_{I_1,\dots I_t} \prod_{j=1}^{t} \left( \sum_{i \in I_j} (b_i + w_i) - 1 \right) p^{b_{I_j}}_{w_{I_j}},$$

# Degenerations in the quadratic case




A joining is admissible if and only if for every $i = 1, \dots, m$ the following holds:

- if all of the descendants of $G_i$ have labels smaller then $i$, then the bridge joining $G_i$ to its parent is in the black corner of $G_i$;
- otherwise, the bridge joining $G_i$ to its parent and the bridge joining $G_i$ to the subtree containing the descendant of $G_i$ of maximal label are in the corners of $G_i$ of different colors.

# Cylinder contributions in $\mathcal{H}(2g-2)$

**Theorem (Y., 2023)**

*The contribution of $n$-cylinder square-tiled surfaces to the volume of $\mathcal{H}(2g-2)$ is equal to $\frac{2(2\pi)^{2g}}{(2g-1)!}a_{g,n}$, where $a_{g,n} \in \mathbb{Q}$, and whose generating function $\mathcal{C}(t,u) = 1 + \sum_{g\geq 1}\left(\sum_{n=1}^{g} a_{g,n}u^n\right)(2g-1)t^{2g}$ satisfies for all $g \geq 0$*

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- Faber-Pandharipande 2000: $\left(\frac{t/2}{\sin(t/2)}\right)^{u+1}$ is the generating function of Hodge integrals $\int_{\overline{\mathcal{M}}_{g,1}} \lambda_{g-i}\psi_1^{2g-2+i}$

# Cylinder contributions in spin components of $\mathcal{H}(2g-2)$

**Theorem (Y., 2024+)**

*The difference of the contributions of $n$-cylinder square-tiled surfaces to the volumes of even and odd spin subspaces of $\mathcal{H}(2g-2)$ is equal to $\frac{2(2\pi i)^{2g}}{(2g-1)!}d_{g,n}$, where $d_{g,n} \in \mathbb{Q}$, and whose generating function $\mathcal{D}(t,u) = 1 + \sum_{g\geq 1}\left(\sum_{n=1}^{g} d_{g,n}u^n\right)(2g-1)t^{2g}$ satisfies for all $k \geq 1$*

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$u = 1$ : formula from Chen, Möller, Sauvaget, Zagier.