Counting metric ribbon graphs

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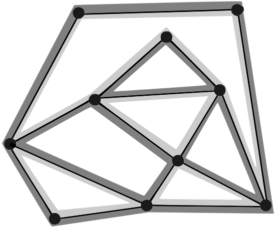
Outline

Metric ribbon graphs

- *Ribbon graph* = combinatorial map
  = graph + circular ordering of edges around every vertex
- A *metric* on a ribbon graph G is simply a function $w : E(G) \to \mathbb{R}_{>0}$.
  A metric w is *integer* if $w : E(G) \to \mathbb{Z}_{>0}$.
- *Length of a boundary / perimeter of a face* = sum of lengths of adjacent edges.
- We will be interested in face-bicolored metric ribbon graphs (every edge is incident to two faces of different colors). These are *hypermaps* from Houcine's talk!.



Counting metric ribbon graphs

Denote by

$$\mathcal{P}_{k,l}^{g,v}(L_1,\ldots,L_k;L'_1,\ldots,L'_l) = \mathcal{P}_{k,l}^{g,v}(L;L')$$

the number of *integer* metric ribbon graphs with:

- genus g;
- vertex degree profile $v = (v_1,\ldots,v_n)$;
- k black (labeled) boundaries of lengths $L = (L_1,\ldots,L_k)$;
- l white (labeled) boundaries of lengths $L' = (L'_1,\ldots,L'_l)$.

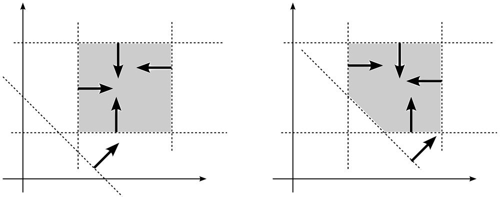
What are the $\mathcal{P}$-functions?

Clearly, $\mathcal{P}^{g,v}_{k,l}(L; L') = 0$ outside of $\mathbb{Z}^{k+l}_{>0} \cap \{L_1 + \ldots + L_k = L'_1 + \ldots + L'_l\}$.

Proposition

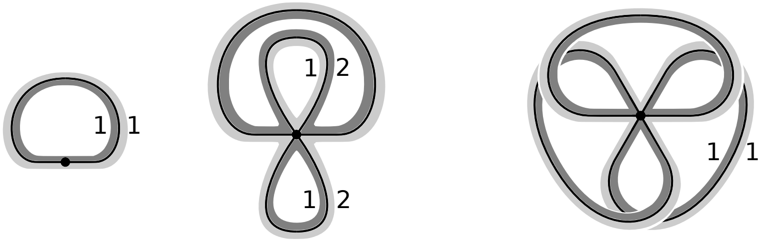
The contribution of each graph to $\mathcal{P}^{g,v}_{k,l}(L; L')$ is a piecewise polynomial function of L, L'. In particular, $\mathcal{P}^{g,v}_{k,l}(L; L')$ is piecewise polynomial.

Follows from the theory of integer points in polyhedra.



Outline

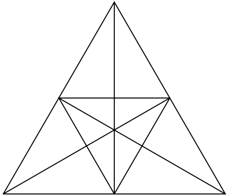
One-vertex graphs



Since there is just one vertex, we omit v from the notation: $\mathcal{P}^{g}_{k,l}(L; L')$.

A result on $\mathcal{P}^g_{k,l}$

Consider the cone $C_{k,l} = \mathbb{R}^{k+l}_{>0} \cap \{L_1 + \ldots + L_k = L'_1 + \ldots + L'_l\}$.
It is sliced by the "walls" given by equations of type $\sum_{i \in I} L_i = \sum_{j \in J} L'_j$.



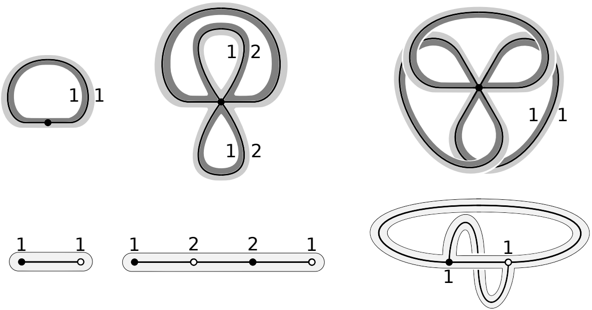
Theorem (Y., 2023)

On $C_{k,l}$ and outside of the walls, $top(\mathcal{P}^g_{k,l})$ is a **polynomial** *of degree $2g$.*

The same statement is true on the intersection of any subset of walls.
The coefficients of all polynomials are explicit.

Proof outline

Pass to the dual ribbon graph:



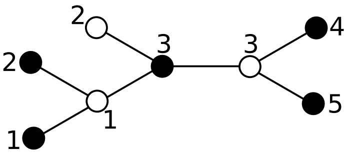
Then $\mathcal{P}^g_{k,l}$ counts metric ribbon graphs with 1 boundary (=*unicellular maps*), vertex-bicolored, with prescribed sums of edge lengths around each vertex.

Case $g = 0$: metric plane trees

Lemma

There is at most one metric on a planar tree with given sums L, L' of edge lengths around every vertex. Moreover, the edge lengths are linear functions of L, L' of the form $\sum_{i \in I} L_i - \sum_{j \in J} L'_j$.

Proof:



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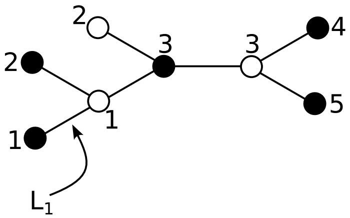

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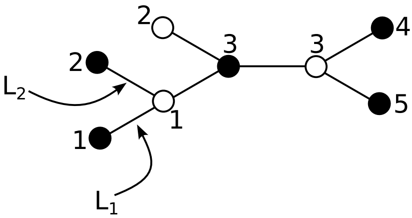

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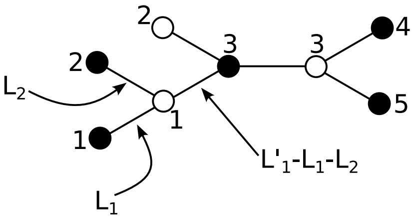

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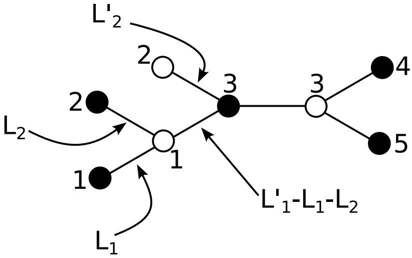

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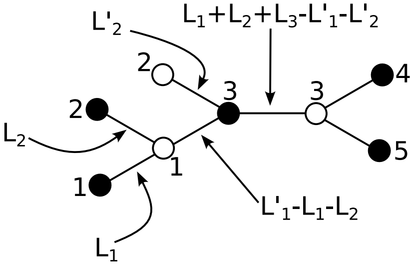


This is a metric $\iff$ all linear forms are positive! □

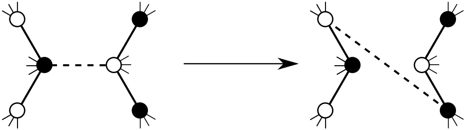
Case $g = 0$: metric plane trees

Lemma $\Rightarrow \mathcal{P}^0_{k,l}(L; L')$ is constant outside of the walls.

Proposition

When the point $(L; L')$ traverses a wall, the value of $\mathcal{P}^0_{k,l}(L; L')$ does not change.

Proof:

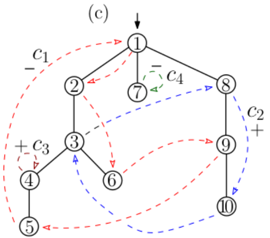


□

Thus $\mathcal{P}^0_{k,l}(L; L')$ is constant outside the walls.

Case $g > 0$: ideas

- There is a bijection (due to Chapuy, Féray, Fusy) between unicellular maps and plane trees decorated with certain permutation on the set of vertices;



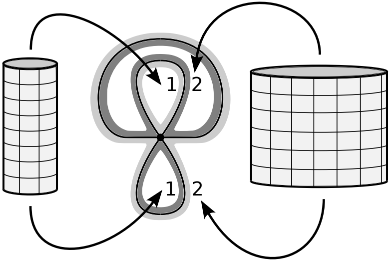
- to get the map, glue vertices in each cycle!
- $top(\mathcal{P}^g_{k,l})$ is the sum of *volumes* of the corresponding polytopes;
- so $top(\mathcal{P}^g_{k,l})$ is the integral of $\mathcal{P}^0_{*,*}$ over simplices $\{x_3 + x_8 + x_{10} = L_2\}$, etc.

□

Outline

A word about motivation

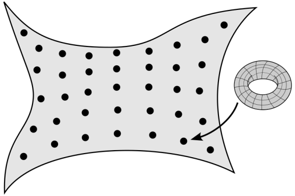
Polynomials $top(\mathcal{P}^g_{k,l})(L;L')$ with $k=l$ and $L_i = L'_i \in \mathbb{Z}$ for all i are useful for *asymptotic enumeration* of *square-tiled surfaces with one singularity*.

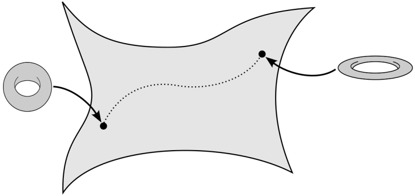


To get a ST surface: for each i, glue a square-tiled cylinder of circumference L_i to the two corresponding boundaries of the ribbon graph.
To come back: canonical decomposition into cylinders + ribbon graph.

A word about motivation

Square-tiled surfaces live inside *continuous* families of flat surfaces.





Ribbon graphs with the same g, k, l can give surfaces from different families (i.e. not connected by a continuous deformation).
How to distinguish ribbon graphs of different types? Do these subfamilies of ribbon graphs enjoy a polynomiality property? $\Rightarrow$ enumerate ST surfaces in each continuous family?

Spin parity

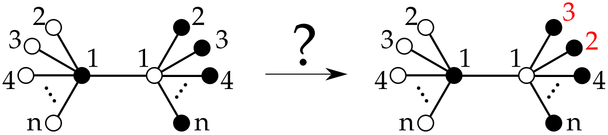
There is a topological invariant of flat surfaces (*spin parity*) which can distinguish different families. What is the corresponding invariant of ribbon graphs? Hopefully, for plane trees this invariant is preserved by flips (otherwise our proof breaks down...)

Theorem (Y.)

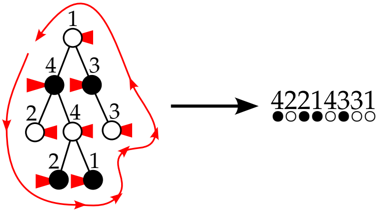
- *If $k - l$ is odd, any two trees with k black and l white labeled vertices are connected by a sequence of flips.*
- *If $k - l$ is even, there are exactly 2 equivalence classes! The counting function for each class is constant outside the walls.*

Spin parity for plane trees

Are these two trees connected by flips?



No! Choose a root, make a tour around your tree, record *first* visits to black vertices and *last* visits to white vertices ("prefix-postfix" traversal). The parity of the obtained permutation is flip-invariant!



Spin parity for $g > 0$

Conjecture (work in progress)

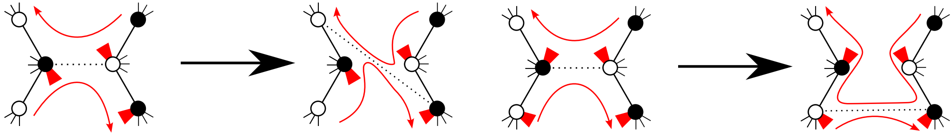
For any g, if $k - l$ is even, there is a partition of the corresponding ribbon graphs into 2 subfamilies such that the counting functions for each subfamily have polynomial top-degree terms.

Outline

Flipping edges cleverly

Claim

- *For any non-root edge in any (rooted) tree, there is a unique way to flip it so that the prefix-postfix sequence does not change!*
- *There is a unique way to flip the root edge so that the prefix-postfix sequence changes by a cyclic permutation.*



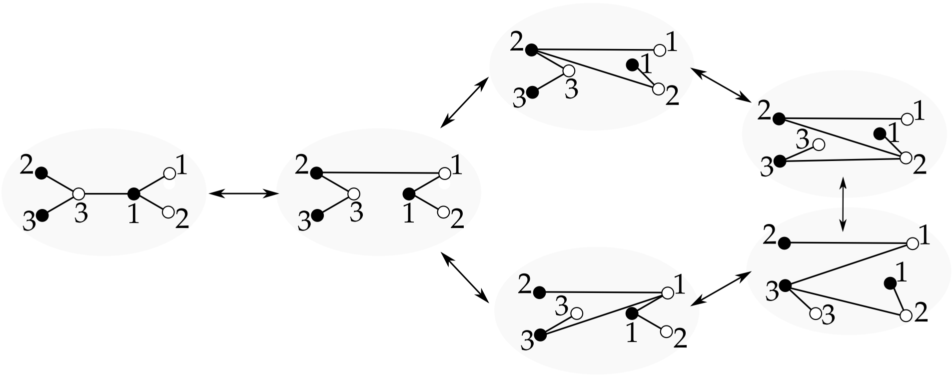
From trees to triangulations

Consider a subfamily F of (rooted) trees with a given prefix-postfix sequence (modulo cyclic permutations).

- The counting function of F is constant. One can show (by a simple counting argument) that it is in fact equal to 1!
- It means that for every point $(L; L')$ there is a unique tree $t \in F$ that contributes at this point.
- But each tree contributes on a simplicial cone (generated by its edges).
- Hence these cones form a simplicial decomposition of the ambient cone C.

Projectivizing, we get a triangulation of the polytope $\mathbb{P}C = \Delta_k \times \Delta_l$ – the product of two simplices!

Example for $k = l = 3$



Outline

Many-vertex graphs

- When there are at least 2 vertices, even the top-degree term of $\mathcal{P}_{k,l}^{g,v}(L;L')$ is not polynomial. However...
- For any g, v, k, l let $\overline{\mathcal{P}}_{k,l}^{g,v}(L;L')$ be the *weighted* sum of contributions of corresponding graphs, where the weight $t(G)$ of a graph G is defined as follows.
- Orient every edge of G in such a way that the black adjacent boundary is on the left. Then $t(G)$ is the number of oriented spanning trees centered at any vertex (this number does not depend on the vertex!).

Conjecture (work in progress)

The top-degree term of $\overline{\mathcal{P}}_{k,l}^{g,v}(L;L')$ *is polynomial for all* g, v, k, l.