Cylinders in square-tiled surfaces of minimal strata

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Journée Cartes à Marne-la-Vallée,
10 March 2023

iyakovlev23.github.io

Outline

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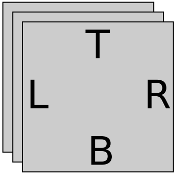
Square-tiled (ST) surfaces



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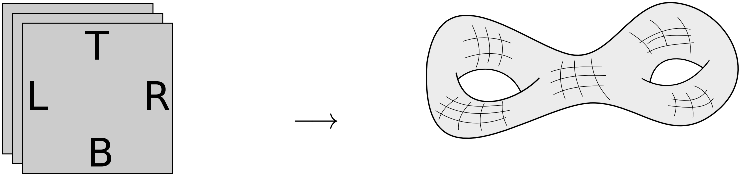

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Oriented, closed surface

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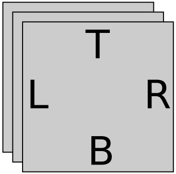

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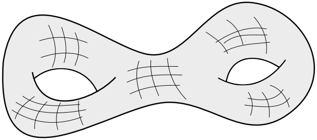

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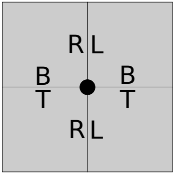

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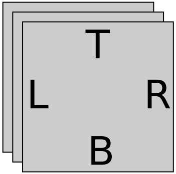

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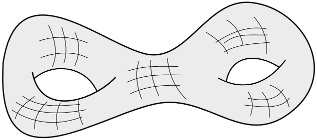

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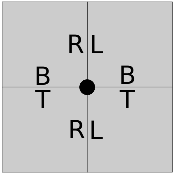

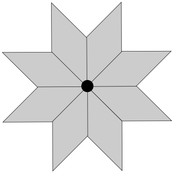

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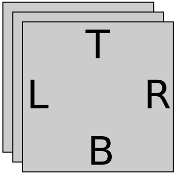

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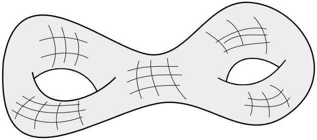

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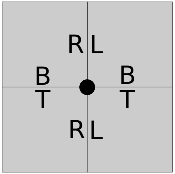

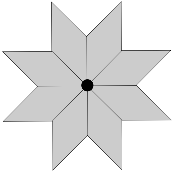

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Square-tiled (ST) surfaces

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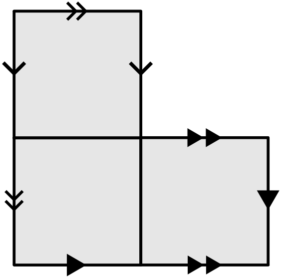

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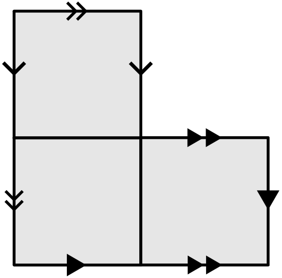

$\Rightarrow$ being a ST surface is a "global" property.

Asymptotic enumeration of ST surfaces

- fix the degrees of vertices which are bigger then 4 ("singularities"):

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- We are interested in the numbers $c(k_1,\ldots,k_s)$.

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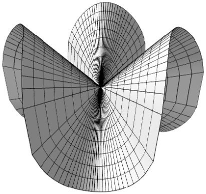

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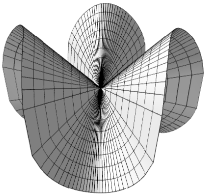
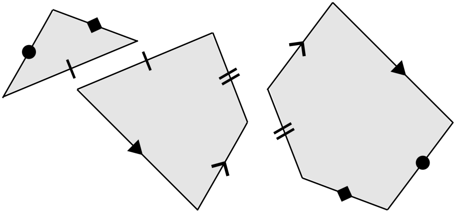
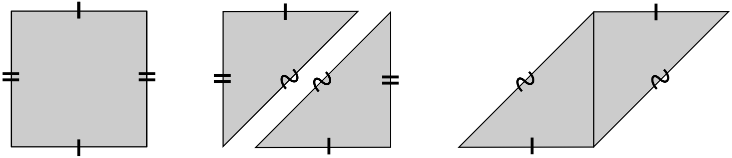




Any translation surface can be obtained by gluing euclidean polygons *along equal and parallel sides*.

Motivation: translation surfaces

NB: a translation surface can be cut into polygons in many different ways!



"Cut and glue"

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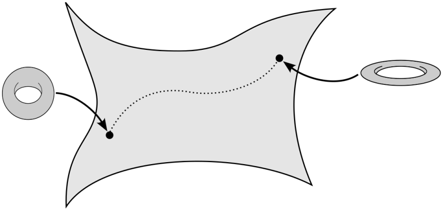

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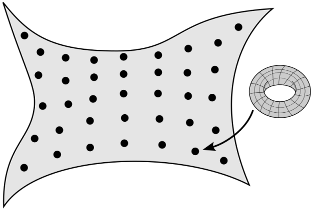

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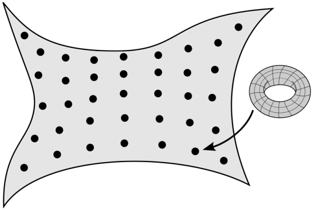

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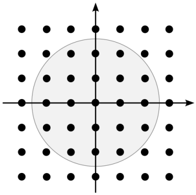

$\mathbb{Z}^n \subset \mathbb{R}^n$

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- Asymptotic enumeration of $\mathcal{ST}(k) \iff$ computing the volume of the "unit ball" in $\mathcal{H}(k)$ for the (natural) *Masur-Veech measure*.

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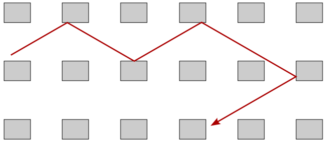

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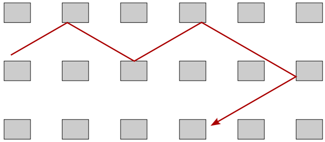

Theorem (Delecroix, Hubert, Lelièvre, 2014)

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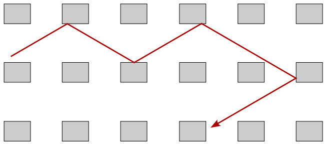

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billiard $\xrightarrow{\text{unfold}}$ translation surface $\xleftrightarrow{\text{renormalization}}$ stratum

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- Sauvaget 2018 (intersection theory) – generating function for $c(2g-2)$;
- Chen, Möller, Sauvaget, Zagier 2020 (intersection theory) – recursion for general strata.

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- Lagrange inversion $\Rightarrow$ explicit formula for $\mathcal{C}(t,u)$ (non-analytic);
- setting $u=1$ we recover the result of Sauvaget.

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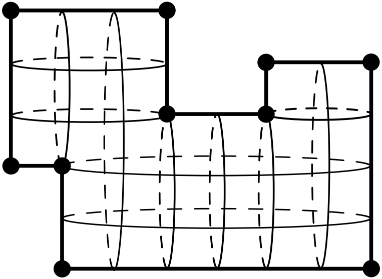
Cylinder decomposition of ST surfaces



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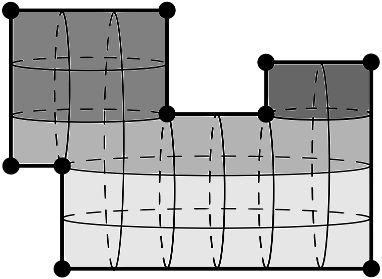

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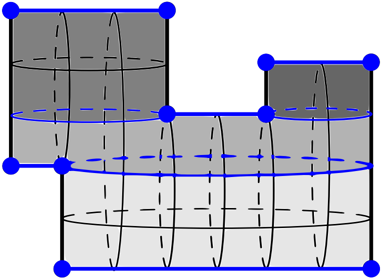

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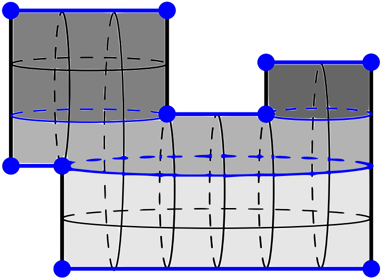

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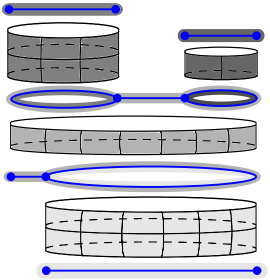

ST surface = cylinders glued along ribbon graphs.

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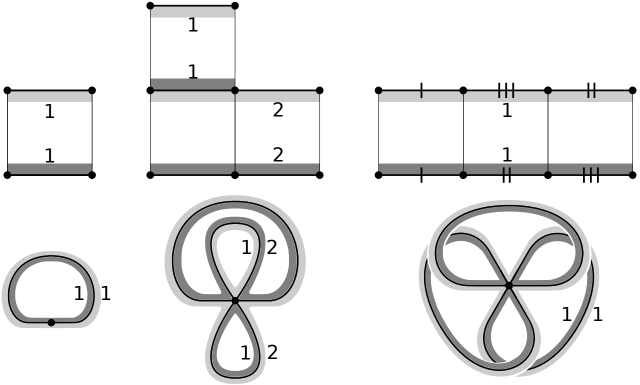
Cylinder decomposition in $\mathcal{H}(2g-2)$



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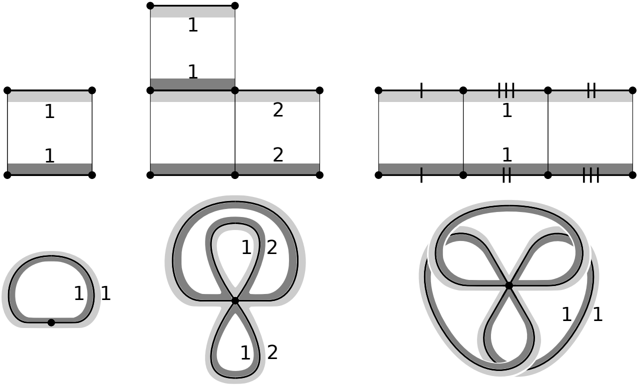

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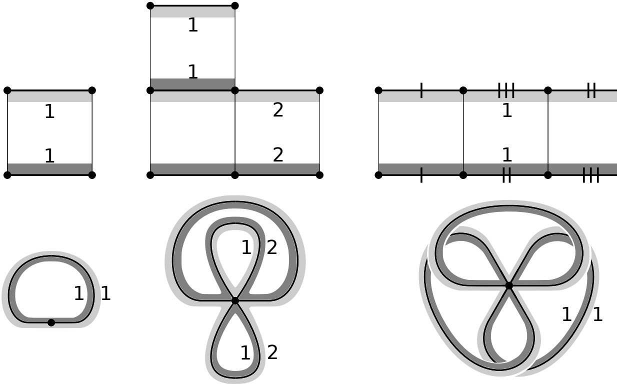

- Number the n cylinders arbitrarily.
- $\Rightarrow$ 1 ribbon graph of genus $g-n$, with 1 vertex, face-bipartite, with n black and n white numbered boundary components

Counting ST surfaces in $\mathcal{H}(2g-2)$

- Let $h_1, \ldots, h_n \in \mathbb{Z}_{>0}$ and $L_1, \ldots, L_n \in \mathbb{Z}_{>0}$ be the heights and the circumferences of the cylinders.

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- Let $h_1, \ldots, h_n \in \mathbb{Z}_{>0}$ and $L_1, \ldots, L_n \in \mathbb{Z}_{>0}$ be the heights and the circumferences of the cylinders.
- Then the number of n-cylinder surfaces in $|\mathcal{ST}_{\leq N}(2g-2)$ is

$$\frac{1}{n!} \cdot \sum_{\substack{\sum_{i=1}^n h_i L_i \leq N \\ h_i, L_i \in \mathbb{Z}_{>0}}} L_1 \cdots L_n \cdot \mathcal{P}_{n,n}^{g-n}(L_1, \ldots, L_n; L_1, \ldots, L_n),$$

where $\mathcal{P}_{n,n}^{g-n}(...)$ is the counting function of *integral metric* ribbon graphs of genus $g-n$, 1 vertex, face-bipartite, n black and n white numbered boundary components of perimeters $L_1, \ldots, L_n$.

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- More generally, let

$$\mathcal{P}^g_{k,l}(L_1, \dots, L_k; L'_1, \dots, L'_l)$$

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- We are interested in *asymptotics*, so everything boils down to studying the *top-degree term* of $\mathcal{P}^g_{k,l}$.

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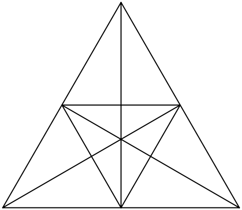
- Every $\mathcal{P}_G(L,L')$ is piecewise polynomial in L, L'. So is, *a priori*, $\mathcal{P}^g_{k,l}(L;L')$.

What are $\mathcal{P}_{k,l}^g$?

Consider the positive cone in the subspace $L_1 + \ldots + L_k = L'_1 + \ldots + L'_l$. Inside, there are "walls" given by equations of type $\sum_{i \in I} L_i = \sum_{j \in J} L'_j$.



What are $\mathcal{P}^g_{k,l}$?

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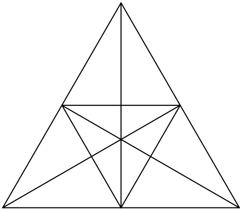

Theorem (Y., 2022)

On the ambient subspace, or on the intersection of any subset of walls, $top(\mathcal{P}^g_{k,l})$ is a **polynomial** *of degree at most $2g$.*
Moreover, its coefficients are the values of $\mathcal{P}^0_{\cdot,\cdot}$ on certain walls, and have a nice combinatorial interpretation.

Case $g = 0$: metric planar trees

Lemma

There is at most one metric on a planar tree with given sums L, L' of edge lengths around every vertex.
Edge lengths are linear functions of L, L' of the form $\sum_{i \in I} L_i - \sum_{j \in J} L'_j$.

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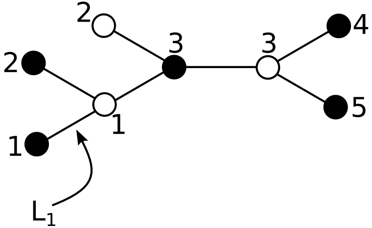

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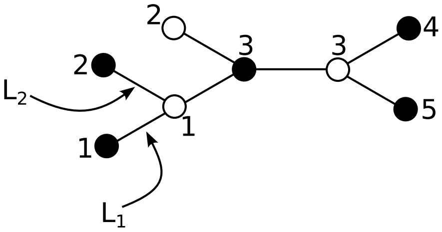

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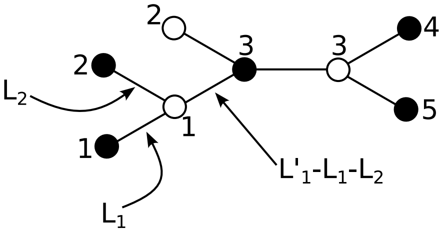

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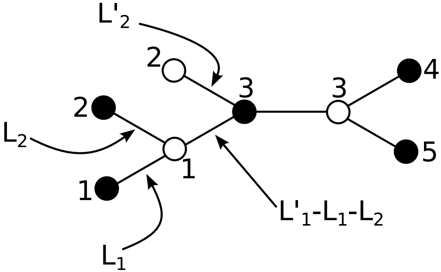

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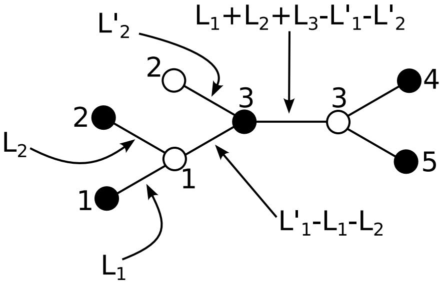

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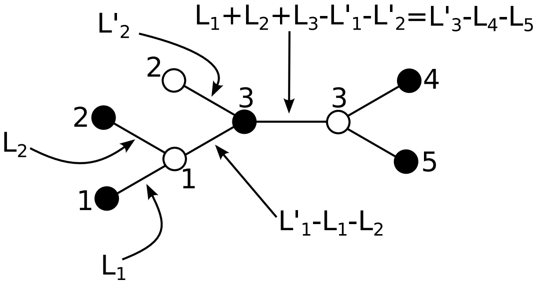

This is a metric only if all linear forms are positive! □

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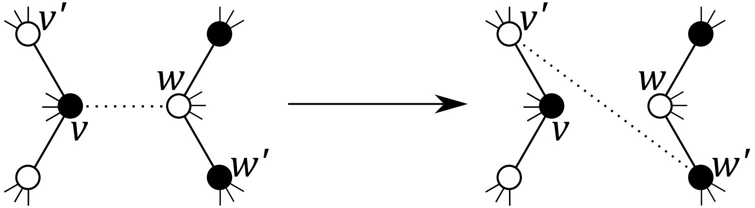

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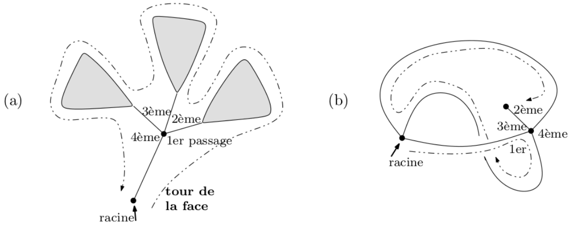

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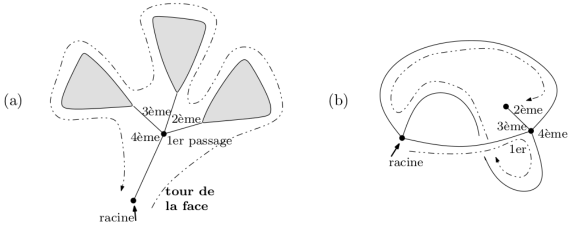

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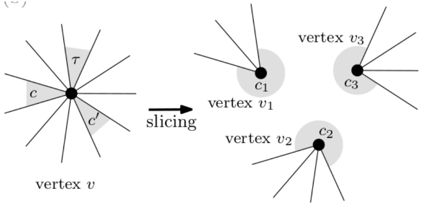

Figure from the paper by Chapuy, Féray, Fusy

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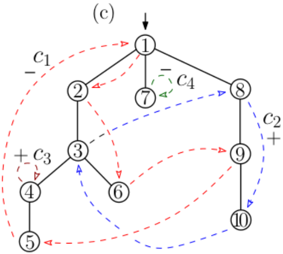

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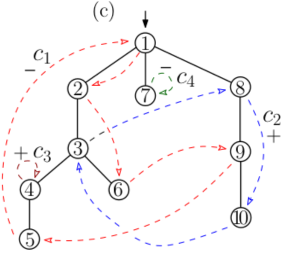

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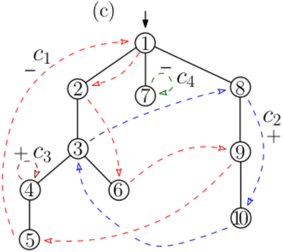

- The bijection is non-explicit and 1-to-many...
- But the *underlying graph* of a unicellular map is equal to the underlying graph of the corresponding tree with vertices in each cycle glued together! And that is enough for us!

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- Since $\mathcal{P}^g_{k,l}$ counts integer points in some polytopes, $\mathrm{top}(\mathcal{P}^g_{k,l})$ computes their *volume*.

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- Summing over all G and T, and using the result for $g = 0$ on the RHS, we see that $\operatorname{top}(\mathcal{P}^g_{k,l})(L; L')$ is a polynomial. □

Upshot

Previous proof also shows that the coefficients of $\mathrm{top}(\mathcal{P}^g_{k,l})$ are values of genus-0 functions $\mathcal{P}^0_{\cdot,\cdot}$ on different intersections of walls.

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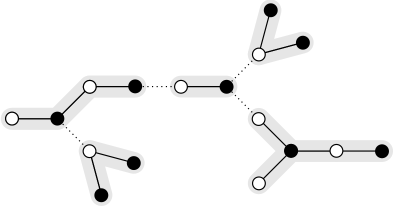

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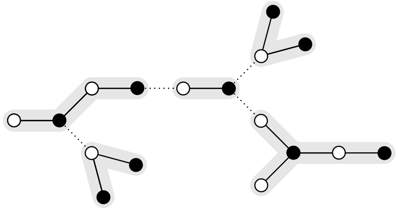

Recursion for values of $\mathcal{P}^0_{k,l}$ $\Rightarrow$ Recursion for the polynomials $\mathrm{top}(\mathcal{P}^g_{k,l})$ $\Rightarrow$ Recursion for the volume contributions $c_{g,n}$.

$\square$

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Inspiration: Kontsevich polynomials

Theorem (Kontsevich, 1992)

Let $L_1 + \cdots + L_n$ be even. The weighted count of **trivalent** *metric ribbon graphs of genus g with n boundaries of perimeters $L_1, \ldots, L_n$ is*

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where $N_{g,n}$ is a homogeneous **polynomial**, *whose* **coefficients are intersection numbers** *of psi-classes on the moduli space of curves $\mathcal{M}_{g,n}$.*

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- Delecroix, Liu, 2022: Distribution of lengths among components.