Contribution of n-cylinder square-tiled surfaces to Masur–Veech volume of $\mathcal{H}(2g-2)$

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Outline

1. Strata of differentials
2. Masur-Veech volumes
3. Cylinders and ribbon graphs
4. Big picture

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- Each stratum carries a natural affine structure, given by the local *period coordinates*: choose a basis $\gamma_1, \ldots, \gamma_{2g+s-1}$ of the relative homology $H_1(C, \{x_1, \ldots, x_s\}; \mathbb{Z})$, where $x_1, \ldots, x_s$ are the zeros of ω; then the coordinates of (C, ω) are

$$\left(\int_{\gamma_i} \omega\right)_{i=1,\ldots,2g+s-1}$$

Transition maps are matrices from $GL(2g + s - 1, \mathbb{Z})$.

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- in particular, flat metric on $C \setminus \{x_1, \ldots, x_s\}$ with *trivial holonomy* and consistent choice of (say) *horizontal direction* at every point;
- how does the metric look like around a zero of ω?

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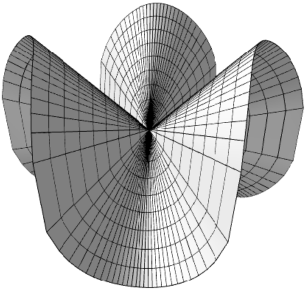

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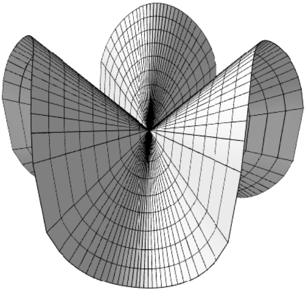

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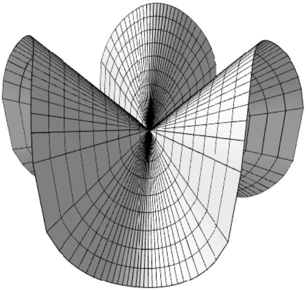

- *conical singularity* of angle $2\pi(k+1)$;
- $k+1$ "horizontal" / "vertical" directions at the singularity.

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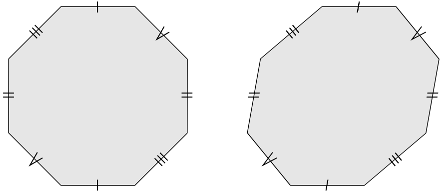

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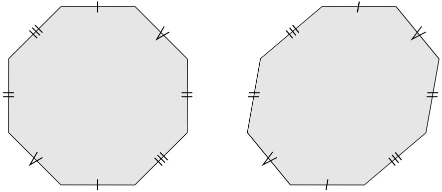

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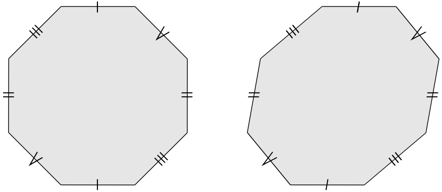
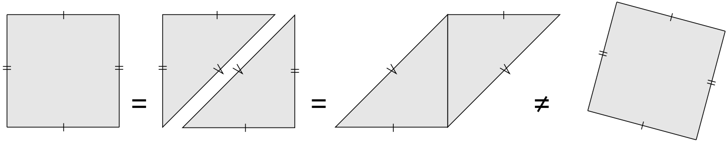


- complex vectors of sides = periods of ω;
- cutting along a diagonal, gluing along identified sides = changing the basis of the relative homology = transition to different period coordinates.



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- exact computation of volumes is important to study the dynamics on the strata: Siegel-Veech constants, Lyapunov exponents...

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- integer points = differentials with periods in $\mathbb{Z} \oplus i\mathbb{Z}$.

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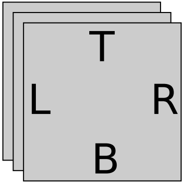

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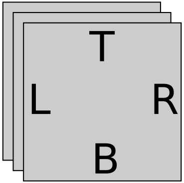

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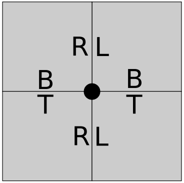

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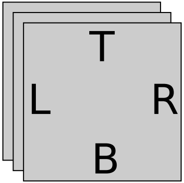

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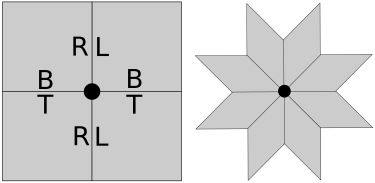

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- Strata have natural affine structure;
- MV measure is Lebesgue measure in the affine charts;
- computing MV volumes is equivalent to asymptotic enumeration of square-tiled (ST) surfaces belonging to the stratum:

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- Chen, Möller, Sauvaget, Zagier (intersection theory) – recursion for general strata.

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Let $\mathrm{Vol}(2g-2) = \frac{2(2\pi)^{2g}}{(2g-1)!} a_g$*, and let* $\mathcal{F}(t) = 1 + \sum_{g \geq 1} a_g (2g-1) t^{2g}$*.*
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Theorem (Y.)

Let the contribution of n-cylinder square-tiled surfaces to $\mathrm{Vol}(2g-2)$ *be equal to* $\frac{2(2\pi)^{2g}}{(2g-1)!} a_{g,n}$, *and let*
$\mathcal{C}(t,u) = 1 + \sum_{g\geq 1} \left(\sum_{n=1}^{g} a_{g,n} u^n\right)(2g-1)t^{2g}$. *Then for all* $g \geq 0$

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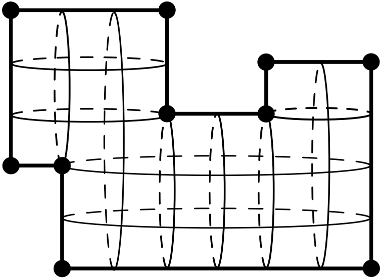

* this is not an Abelian differential...

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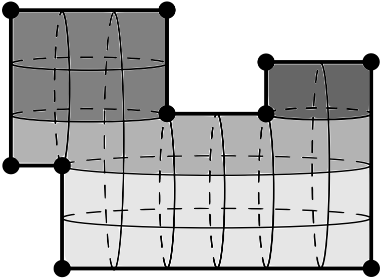

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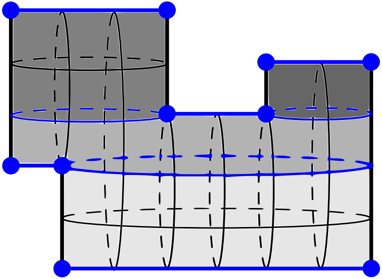

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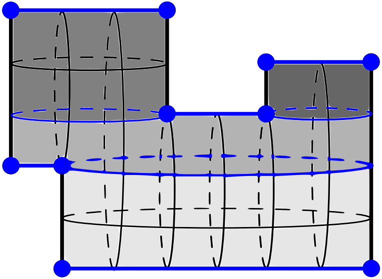

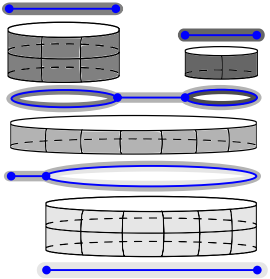

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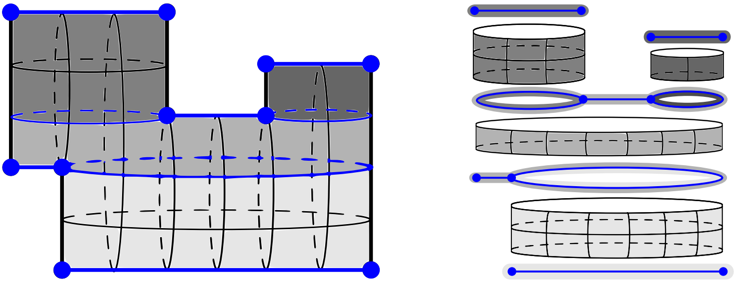

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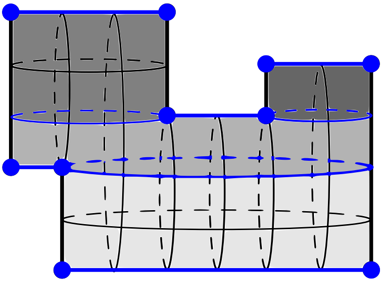

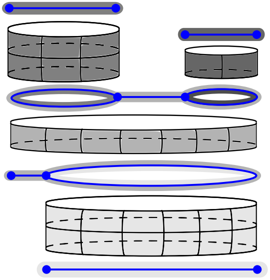

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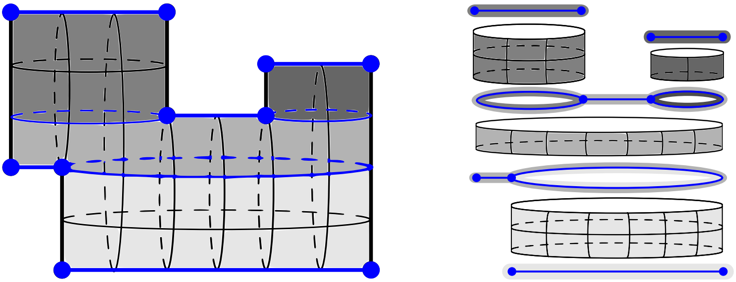

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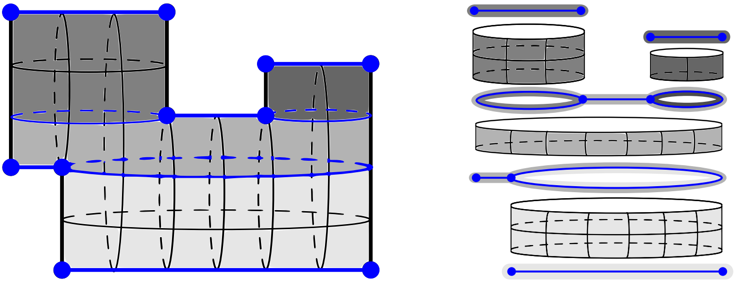

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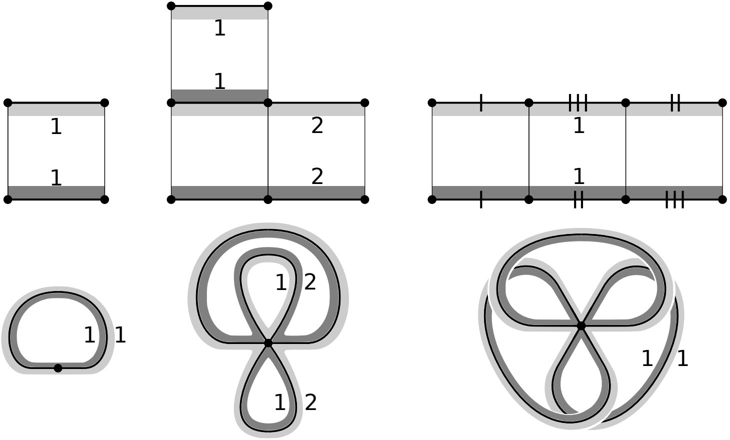
- cylinders are glued to the boundary components of ribbon graphs.

Cylinder decomposition in $\mathcal{H}(2g-2)$



n cylinders $\Rightarrow$ 1 ribbon graph of genus $g-n$, with 1 vertex, $2n$ boundary components of 2 colors, adjacent boundaries have different colors.

Counting square-tiled surfaces

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Let $h_1, \dots, h_n \in \mathbb{Z}_{>0}$ and $L_1, \dots, L_n \in \mathbb{Z}_{>0}$ be the heights and the circumferences of the cylinders.
Then the number $|\mathcal{ST}_n(\mathcal{H}(2g-2), N)|$ of n-cylinder square-tiled surfaces in $\mathcal{H}(2g-2)$ with at most N squares is equal to

$$\frac{1}{n!} \cdot \sum_{\substack{\sum_{i=1}^n h_i L_i \leq N \\ h_i, L_i \in \mathbb{Z}_{>0}}} L_1 \cdots L_n \cdot \mathcal{P}_{n,n}^{g-n}(L_1, \dots, L_n; L_1, \dots, L_n),$$

where $\mathcal{P}_{n,n}^{g-n}(...)$ is the counting function of *integral metric* ribbon graphs of genus $g-n$, 1 vertex, n black and n white boundary components of perimeters $L_1, \dots, L_n$.

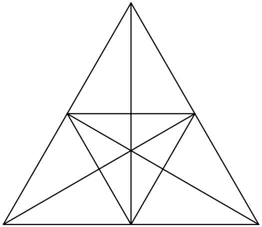
Counting ribbon graphs

More generally, let

$$\mathcal{P}^g_{k,l}(L_1, \ldots, L_k; L'_1, \ldots, L'_l)$$

be the counting function for the *integral metric* ribbon graphs of genus g, with 1 vertex, k black and l white boundary components of perimeter $L_1, \ldots, L_k$ and $L'_1, \ldots, L'_l$ respectively.

Counting ribbon graphs



- Piecewise polynomial function of L, L', of degree $2g$;

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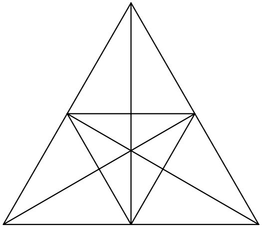

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Theorem (Y.)

On the generic subspace, or on the intersection of any number of walls, $top(\mathcal{P}^g_{k,l})$ is a polynomial. Moreover, its coefficients are the values of $\mathcal{P}^0_{\cdot,\cdot}$ on certain walls.

Outline

Big picture

"Proof" of Main Theorem:

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- $top(\mathcal{P}^g_{k,l})$ are analogous to Kontsevich polynomials, whose coefficients are intersections of psi classes on $\overline{\mathcal{M}}_{g,n}$;

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- $top(\mathcal{P}^g_{k,l})$ are analogous to Kontsevich polynomials, whose coefficients are intersections of psi classes on $\overline{\mathcal{M}}_{g,n}$;
- $\mathcal{P}^g_{k,l}(L;L')$ are the double Hurwitz numbers with a single cycle $H^{\mathbb{C}P^1}(L;(2g+k+l-1);L')$;

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- $\mathcal{P}^g_{k,l}(L; L')$ are the double Hurwitz numbers with a single cycle $H^{\mathbb{C}P^1}(L; (2g+k+l-1); L')$;
- intersection-theoretic interpretation of $a_{g,n}$?