HEAT KERNEL ON COMPACT MANIFOLDS

0.1. **Existence and asymptotic expansion.** *Reference*: Rosenberg Chapter 3.

Recall that the heat kernel of $\mathbb{R}^n$ is

$$p(x,y,t) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}.$$

To guess this formula, consider $u(x,t)$ solution of the heat equation with initial condition $f(x)$. Then the Fourier transform $\hat{u}(\xi,t)$ satisfies $(\partial/\partial t)\hat{u}(\xi,t) = -|\xi|^2\hat{u}(\xi,t)$ with initial condition $\hat{f}(\xi)$. Hence $\hat{u}(\xi,t) = \hat{f}(\xi)e^{-|\xi|^2 t}$. Taking inverse Fourier transform gives that u is the convolution of f with p.

We now construct the heat kernel $p_M(x,y,t)$ on an arbitrary closed compact Riemannian manifold M. In particular, it will satisfy the following *Manikshisundaram–Pleijel asymptotic expansion*:

$$p(x,y,t) \sim (4\pi t)^{-n/2} e^{-d(x,y)^2/4t}(u_0 + u_1 t + \ldots),\ t \to 0, \tag{1}$$

where $u_i \in C^\infty(M \times M)$.

The idea is to have an ansatz for the heat kernel as in (1). That is we assume that p is equal (formally) to that infinite series, even though in the end this will only be an *asymptotic* expansion. Let $\varepsilon < \mathrm{inj}(M)$. For now we only consider x, y in $U_\varepsilon = \{(x,y) \in M \times M \mid\ : d(x,y) < \varepsilon\}$.

Denote $L = \partial/\partial t + \Delta_x$. We know that $Lp = 0$. Now apply this operator to the right hand side to find the unknown functions u_i. For fixed y, this is best done in exponential polar coordinates for x centered around y. Let $r = d(x,y)$. Let (g_{ij}) be the metric in polar coordinates and $g = det(g_{ij})$. Using formula for the Laplacian in polar coordinates of a radial function

$$\Delta f = -\frac{\partial^2 f}{\partial r^2} - \frac{\partial f}{\partial r}\left(\frac{n-1}{r} + \frac{\partial_r\sqrt{g}}{\sqrt{g}}\right)$$

and setting the coefficients of all powers of t to zero, we get the following equations for the u_i:

$$\partial_r u_0/u_0 = -\frac{1}{2}\partial_r\sqrt{g}/\sqrt{g} \tag{2}$$

$$\partial_r u_i + (\frac{1}{2}\partial_r\sqrt{g}/\sqrt{g} + \frac{i}{r})u_i + \frac{1}{r}\Delta_x u_{i-1} = 0 \tag{3}$$

First equation is equivalent to $\partial_r \ln(u_0) = -\frac{1}{2}\partial_r \ln(\sqrt{g})$, hence $u_0(x,y) = (\sqrt{g})^{-1/2}(x,y)$, where we have chosen the constant of integration equal to 1 (this is to ensure that the H_k defined later is a delta approximation). In particular, $u_0(x,x) = 1$.

Assume $u_0, \ldots, u_{i-1}$ have been found and are C^∞. We use the second equation to find u_i recursively. The homogeneous version of the second equation has solution of the form $Kr^{-i}(\sqrt{g})^{-1/2}$, where K only depends on angular coordinates. Using variation of the constant $K = K(r)$, we find from the full equation that $\partial_r K(r) = -\sqrt{g}^{-1/2}\Delta_x u_{i-1} r^{i-1}$, and we find K by integration, with $K(0) = 0$. In order for u_i to be C^∞ we need $K(r) = O(r^i), r \to 0$, which can be easily checked. Note in particular that for $(x,y)\in U_\varepsilon$ the values of u_i are uniquely determined by the metric.

Denote $G(x,y,t)=(4\pi t)^{-n/2}e^{-d(x,y)^2/4t}$. Now consider the (well-defined) function

$$H_k(x,y,t)=G(x,y,t)(u_0+u_1t+\ldots+u_kt^k),$$

and extend it to $M\times M$ by smoothing it out to zero outside of U_ε. By construction we have $LH_k=Gt^k\Delta_x u_k$ on U_ε, which is close to zero for small t. Moreover, H_k is still an approximation of the delta function $\delta_y(x)$ when $t\to 0$ (because $u_0(x,x)=1$). Hence H_k is a good approximation of the heat kernel. We now promote it to a full heat kernel.

Let X,Y operators on a Hilbert space, and assume they have well defined *heat operators* e^{-tX},e^{-tY}, i.e. a semigroup of bounded self-adjoint operators satisfying

$$(\partial_t+X)e^{-tX}=0,\ \forall f:\lim_{t\to 0}e^{-tX}f=f.$$

Suppose we know e^{tX} and Y and would like to compute $e^{-t(X+Y)}$.

Proposition 1 (Duhamel's formula)**.** *Provided $e^{-t(X+Y)}$ exists, we have*

$$e^{-t(X+Y)}=e^{-tX}-e^{-t(X+Y)}*(Ye^{-tX}),$$

where $*$ is the convolution $A*B=\int_0^t A(t-s)B(s)ds$.

Proof. Set $B(t)=e^{-t(X+Y)}e^{tX}$. Then $dB/dt=-e^{-t(X+Y)}Ye^{tX}$. Now integrate from 0 to t and multiply by e^{-tX}. □

One can iterate Duhamel's formula and hope that

$$e^{-t(X+Y)}=e^{-tX}+e^{-tX}*\sum_{i=1}^{\infty}(-1)^i(Ye^{-tX})^{*i}.$$

We *assume* that there exists an operator X such that e^{-tX} is the integral operator with kernel H_k. In particular $\partial_t H_k=-XH_k$. We let $Y=\Delta-X$. Then on the level of kernels, the previous formula gives

$$\begin{aligned}p&=H_k+H_k*\sum_{i=1}^{\infty}(-1)^i((\Delta-X)H_k)^{*i}=H_k+H_k*\sum_{i=1}^{\infty}(-1)^i((\Delta+\partial_t)H_k)^{*i}\\&=H_k+H_k*\sum_{i=1}^{\infty}(-1)^i(LH_k)^{*i},\end{aligned}$$

where the convolution of kernels is defined as

$$(F*G)(x,y,t)=\int_0^t d\tau\int_M F(x,z,t-\tau)G(z,y,\tau)dz.$$

We now check that p is well defined and is the heat kernel.

Since $LH_k=Gt^k\Delta_x u_k$, for $k>n/2$ LH_k extends by zero continuously to $t=0$ on U_ε; outside of U_ε it is already zero; in the smoothing region in between, $d(x,y)$ is bounded from below, and so the exponential wins and goes to zero as well. One shows similarly that for $k>n/2+l$ LH_k extends C^l to $t=0$.

One can easily show that the series is absolutely convergent. Moreover, for any T, this series is $O(t^{k-n/2}), t\in[0,T]$. Its convolution with H_k is well defined as well: the only thing to check is the convergence of the integral over τ at $\tau=t$, but this is fine, because H_k is a delta-approximation. Similarly one shows that the convolution is in fact C^l for $k > n/2 + l$.

Note that $L(H_k * G) = G + LH_k * G$. We compute

$$Lp = LH_k + \sum_{i=1}^{\infty}(-1)^i (LH_k)^{*i} + LH_k * (\sum_{i=1}^{\infty}(-1)^i (LH_k)^{*i}) = 0.$$

Finally, H_k is a delta-approximation, and the second term is $O(t^{k-n/2}), t \to 0$, so p is a delta-approximation. Hence p is the heat kernel.

By uniqueness of the heat kernel, we obtain the same p for any $k > n/2$. Hence $p \in C^\infty$ and it satisfies the asymptotic expansion (1).

0.2. **Applications.** *Reference*: BGM III.E.

Take $x = y$ in (1) and integrate over M. One gets the asymptotic expansion

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \sim (4\pi t)^{-n/2}(a_0 + a_1 t + \ldots),\ t \to 0, \tag{4}$$

where $a_i = \int_M u_i$.

Recall that $u_0(x,y) = (\sqrt{g})^{-1/2}(x,y)$. In particular, $u_0(x,x) = 1$, since (g_{ij}) is the metric in normal coordinates around x. Thus

$$a_0 = \mathrm{Vol}(M).$$

How can one express $u_i(x,x)$ for $i \geq 1$ in terms of the metric? By a theorem of Elie Cartan, the metric g_{ij} in exponential coordinates has a Taylor expansion at 0 whose coefficients are (universal) polynomials in the curvature tensor of M at x and its covariant derivatives. Because of the way we constructed the u_i (equations (2), (3)), the same turns out to be true for the u_i. By scaling the metric and looking at some particular manifolds (like spheres or products), one can deduce for example:

$$a_1 = \frac{1}{6}\int_M S\ dV, \tag{5}$$

$$a_2 = \frac{1}{360}\int_M (2|R|^2 - 2|Ric|^2 + 5S)\ dV, \tag{6}$$

where R is the curvature tensor, Ric is the Ricci tensor and S is the scalar curvature.

We draw the following conclusions:

1. spectrum of the Laplacian determines the dimension and the volume;
2. isospectral Riemannian surfaces have the same genus; (a_1 and Gauss-Bonnet);
3. a manifold isospectral to a surface of constant curvature is a surface of same genus and with same constant curvature;
   (in dimension 2, $|R|^2 = S^2, |Ric|^2 = S^2/2$, so from a_2 we get equality of $\int S^2$; then use Cauchy-Schwartz: $(\int S)^2 \leq (\int S^2) \cdot (\int 1)$; the equality appears iff $S = const$; this happens for the surface $\Rightarrow$ this happens for the unknown manifold as well);
4. similar result in dimension 3 ($|R|^2 \geq 2/(n-1)|S|^2, |Ric|^2 \geq S^2/n$, with equality iff constant sectional curvature / $Ric = const \cdot g$ repsectively; also in dim 3 we have $S^2 - 4|Ric|^2 + |R|^2 = 0$);
5. in dim 4, to get the same result, one adds the condition $\chi(M) = \chi(M')$; by Gauss-Bonnet-Chern $\int(|R|^2 - 4|Ric|^2 + S^2) = 32\pi^2\chi$.

Finally, Karamata's Tauberian theorem and (4) imply *Weyl's law*:

$$\#\{i : \lambda_i(M) \leq s\} \sim \frac{\Gamma(\frac{n}{2}+1)\operatorname{Vol}(M)}{(4\pi)^{n/2}} s^{n/2}, n \to \infty.$$

Theorem 1 (Karamata's Tauberian theorem)**.** *If* $\omega(s) = \int_0^\infty e^{-st} dF(t)$ *then*

$$\omega(s) \sim Cs^{-\rho} \iff F(t) \sim \frac{C}{\Gamma(\rho+1)} t^{\rho}.$$